

Space-time extensions II

István Rácz *

RMKI, H-1121 Budapest, Konkoly Thege Miklós út 29-33.

Hungary

June 29, 2010

Abstract

The global extendibility of smooth causal geodesically incomplete spacetimes is investigated. Denote by γ one of the incomplete non-extendible causal geodesics of a causal geodesically incomplete spacetime (M, g_{ab}) . First, it is shown that it is always possible to select a synchronised family of causal geodesics Γ and an open neighbourhood \mathcal{U} of a final segment of γ in M such that \mathcal{U} is comprised by members of Γ , and suitable local coordinates can be defined everywhere on \mathcal{U} provided that γ does not terminate either on a tidal force tensor singularity or on a topological singularity. It is also shown that if, in addition, the spacetime, (M, g_{ab}) , is globally hyperbolic, and the components of the curvature tensor, and its covariant derivatives up to order $k - 1$ are bounded on \mathcal{U} , and also the line integrals of the components of the k^{th} -order covariant derivatives are finite along the members of Γ —where all the components are meant to be registered with respect to a synchronised frame field on \mathcal{U} —then there exists a C^{k-} extension $\Phi : (M, g_{ab}) \rightarrow (\widehat{M}, \widehat{g}_{ab})$ so that for each $\bar{\gamma} \in \Gamma$, which is inextendible in (M, g_{ab}) , the image, $\Phi \circ \bar{\gamma}$, is extendible in $(\widehat{M}, \widehat{g}_{ab})$. Finally, it is also proved that whenever γ does terminate on a topological singularity (M, g_{ab}) cannot be generic.

PACS number: 04.20.Cv, 04.20.Dw, 02.40.Vh

1 Introduction

In Einstein's theory of gravity a spacetime is supposed to be represented by a pair (M, g_{ab}) ¹, where M is a smooth, Hausdorff, paracompact, connected, orientable manifold endowed with a smooth Lorentzian metric g_{ab} [29, 61]. What is more important it is always assumed implicitly that M represents all the events compatible with the history of the investigated physical system. Thereby, for long there had seemed to be no reason to look for spacetime extensions.

*email: iracz@rmki.kfki.hu

¹Throughout this paper, M is assumed to be of arbitrary dimension, $n \geq 2$, the signature of g_{ab} is chosen to be $(-, +, \dots, +)$. Moreover, it is supposed that (M, g_{ab}) is time orientable and that a time orientation has been chosen.

The simplest possible context in which spacetime extensions showed up and did, in fact, play important role was related to coordinate singularities. It took considerably long time to get the maximal analytic extension of the Schwarzschild solution by Kruskal [35] and Szekeres [54]. Even after being panoplied with the acquired technical experiences, it was not obvious at all neither to get the maximal analytic extension of the Kerr spacetime, which was given by Boyer and Lindquist [8], nor to understand its global structure [9]. Many of the results of these pioneering investigations, concerning the extendibility of static or stationary-axisymmetric black hole configurations in Einstein's theory of gravity, were later generalised in various directions (see, e.g., Refs. [62, 41, 43, 20, 44]). In particular, by making use of the techniques of spacetime extensions, it was justified that the event horizon of static or stationary-axisymmetric non-degenerate black hole spacetimes does possess a bifurcate horizon structure in any covariant metric theory of gravity. Other, not directly related areas where, by the introduction of new coordinates, the basic techniques of spacetime extensions could also be applied are the investigation of the class of spacetimes possessing a compact Cauchy horizon [20, 44] or the study of isotropic cosmological singularities [55, 56, 57, 58].

However significant the above mentioned developments are, the most important results claiming for a much better understanding of the various possible concepts of incompleteness and whence for that of the extendibility of spacetimes was manifested by the series of theorems, now referred as singularity theorems, by Penrose [38, 39] and Hawking [30, 31] (see also [21, 29, 61, 2]). These theorems justified that it is a general feature of spacetimes describing the expanding universe and the gravitational collapse of stars that they are causal geodesically incomplete, i.e., they contain non-complete and non-extendible causal geodesics. One may question whether the mere existence of these incomplete causal geodesics has any relevance in physics. In this respect it is worth keeping in mind that timelike geodesics are supposed to represent histories of freely falling test particles or observers. Hence the incompleteness of these type of geodesics implies that the entire history of the corresponding particles or observers can be represented by world-lines with finite intervals of proper time in a spacetime which, on the other hand, is supposed to consist of all the events compatible with the associated physical system.

In spite of the considerable efforts and many of the partial successes (see, e.g., [22, 23, 24, 25, 29, 19, 61, 2] for associated details) there have also been many failures in trying to get an appropriate understanding of the essence of spacetime singularities and in the proper handling of them. The corresponding deficiencies did claim for a sort of creativity which yielded an abundance of spacetimes with non-physical or quasi-regular singularities.² An excellent review on this subject was given by Ellis and Schmidt [18] where plenty of examples with artificial singularities, i.e., geodesically incomplete spacetimes produced by certain “cutting and gluing” processes, are presented. With the aim of providing a generic result on the extendibility of geodesically incomplete spacetimes it is inevitable to face with the existence of these sort of spacetime models. Without getting into the details here, let us merely mention that not all the examples with quasi-regular singularities can

²In case of a true physical or geometrical singularity the blow up of certain curvature tensor components is expected to happen “at the ideal end” of the pertinent incomplete causal geodesic. As opposed to this, in case of a quasi-regular singularity such a blow up never occurs.

be ruled out simply by requiring the spacetimes to be, say, globally hyperbolic (see, e.g., Example II. of [11]). As we shall see below, some of the results of the present paper (see Section 5) justify the validity of the intuitive expectation that whenever a globally hyperbolic spacetime does possess a quasi-regular or, as it will be referred here, *topological* singularity (see Definition 3.2) the spacetime geometry has to be special.

The simplest possible concept of a spacetime extension, that will also be applied in this paper, concerns only the core of the underlying mathematical structures and it can be formulated as follows (see also, [29]). Consider two spacetimes (M, g_{ab}) and $(\widehat{M}, \widehat{g}_{ab})$ the differential structure of which are at least of class C^X , respectively. The map $\Phi : (M, g_{ab}) \rightarrow (\widehat{M}, \widehat{g}_{ab})$ is said to be a C^X -isometric imbedding if Φ is a C^X -diffeomorphism between M and $\Phi[M] \subset \widehat{M}$, and also that the derivative Φ^* of Φ carries the metric g_{ab} into $\widehat{g}_{ab}|_{\Phi[M]}$, i.e., $\Phi^*g_{ab} = \widehat{g}_{ab}|_{\Phi[M]}$. Then, $(\widehat{M}, \widehat{g}_{ab})$ is called to be a C^X -extension of (M, g_{ab}) if $\Phi[M]$ is a proper subset of \widehat{M} . Otherwise (M, g_{ab}) and $(\widehat{M}, \widehat{g}_{ab})$ are considered to be equivalent C^X -representations of the same spacetime.

Notice that in the above definition the differentiability class of the involved spacetimes had been left to be pretty flexible. First of all, the differentiability class of (M, g_{ab}) was not required to be exactly C^X , i.e., (M, g_{ab}) may belong to any higher differentiability class. Second, in the above definition it was not specified either whether only the metric structure or some other fields, as well, are required to be of certain differentiability class. Indeed, in particular cases, beside restricting the differentiability class of the metric structure, one may also want to impose conditions on the differentiability properties of other fields such as certain part of the curvature or some of the involved matter fields. This sort of combined specification was applied, e.g., by Clarke in defining the class of $C^{0-\alpha}$ spacetimes in [14], where the differentiability classes of both the metric and the curvature were restricted respectively.

Motivated mainly by the implications of the singularity theorems, the first systematic investigation of the existence of spacetime extensions, in a generic context, was initiated and carried out by Clarke [11, 13, 14, 15]. He considered both the local and global extendibility of causal geodesically incomplete spacetimes. His main result asserts that, for a generic globally hyperbolic causal geodesically incomplete spacetime there is a $C^{0-\alpha}$ global extension if the Riemann tensor is Hölder-continuous³, i.e., if the original spacetime itself is of class $C^{0-\alpha}$. We would like to emphasise that in spite of the unquestionable significance of these pioneering investigations there are certain unsatisfactory aspects of them. Firstly, Clarke's results are based on the use of the b -boundary construction which, on the other hand, is known to have serious defects even for the simplest Friedman-Robertson-Walker cosmological

³Let \mathcal{O} be an open subset of \mathbb{R}^n . Then, a function $f : \mathcal{O} \rightarrow \mathbb{R}$ is Hölder-continuous with exponent $\alpha \in (0, 1)$ if for each open subset $\mathcal{U} \subset \mathcal{O}$ with compact closure in \mathcal{O} there exist $K > 0$ such that for each pair of points $x, y \in \mathcal{U}$ we have that $|f(x) - f(y)| < K \cdot |x - y|^\alpha$, where $|x - y| = \sqrt{\sum_{i=1}^n (x^i - y^i)^2}$. The class of Hölder-continuous functions with exponent α is denoted by $C^{\alpha-}$. Notice that whenever the above definition holds with $\alpha = 1$ the function f is called to be locally Lipschitz whereas the set of locally Lipschitz functions is denoted by C^{1-} .

model (see, e.g., [33, 53]). This is a substantial disadvantage because even the condition that the spacetime is of class $\mathcal{C}^{0,\alpha}$ cannot be spelled out properly without making use of the concept of b -boundary (for more details see section 5.2 of [14]). Another unsatisfactory aspect is that the above recalled result allows the possibility that in a spacetime which cannot be extended within the class $\mathcal{C}^{0,\alpha}$ simply the curvature may fail to be Hölder-continuous without becoming unbounded somewhere. This, however, indicates that Clarke's results are not sharp enough to be applied in the justification of the idea that, in a generic spacetime, the existence of an incomplete inextendible causal geodesic has to be associated with a true physical singularity.

The present work is our second paper devoted to the study of the extendibility of spacetimes containing incomplete inextendible causal geodesics. Previously in [42], sufficient conditions were given ensuring the existence of local extensions of causal geodesically incomplete spacetimes. Here, by making use of some of the results already established in [42], along with several newly derived ones, the existence of *global* extensions to smooth (C^∞) causal geodesically incomplete spacetimes will be investigated. According to our main result to any smooth *generic*⁴ globally hyperbolic causal geodesically incomplete spacetime (M, g_{ab}) , say with an incomplete causal geodesic γ , there exists a C^{k-} extension $\Phi : (M, g_{ab}) \rightarrow (\widehat{M}, \widehat{g}_{ab})$, i.e., \widehat{g}_{ab} is of class C^{k-} , whenever the components of the curvature tensor, and that of its covariant derivatives up to order $k-1$ are guaranteed to be bounded and also the line integrals of the components of the k^{th} -order covariant derivatives are finite along the members of an $n-1$ -parameter congruence of causal geodesics Γ in a sufficiently small open neighbourhood of a final segment of $\gamma \in \Gamma$, where all the components are meant to be measured with respect to a synchronised frame field. It is also shown that for those causal geodesics $\bar{\gamma} \in \Gamma$ which are sufficiently close to γ and are also incomplete and inextendible in (M, g_{ab}) the causal geodesics $\Phi \circ \bar{\gamma}$ can be extended in $(\widehat{M}, \widehat{g}_{ab})$.

Before turning to the more technical issues we would like to make a clear distinction between the type of conditions imposed by Clarke in his approach and that of the conditions applied in our work. The most significant difference is conceptual and, essentially, it is rooted in the facts that in our approach we do insist on using only those structures that are provided by the spacetime itself, and also that throughout the associated investigations, as opposed to Clarke's approach, all of our constructions make explicit use of the Lorentzian character of the spacetime metric. To provide a clear manifestation of the related differences recall that, for instance, to properly spell out the conditions used by Clarke one needs to consider neighbourhoods of an "ideal endpoint" of a horizontal lift of the selected incomplete geodesic in the linear frame bundle. This ideal endpoint does correspond to a point on the b -boundary. Accordingly, it is needed to be checked whether the curvature tensor components satisfy the Hölder condition in a sufficiently small neighbourhood of such a boundary point, which itself is not a 'lift' of a regular spacetime point and its neighbourhoods do not make sense without defining them within the framework of the b -boundary construction, based on the use of a non-physical Riemannian metric on the linear frame bundle. As opposed to Clarke's approach, in [42] and also in this paper, the conditions we apply refer only to the

⁴The meaning of genericness applied here will be made clear later.

behaviour of certain physical quantities (like the tidal force tensor components) which could be measured by a family of real observers, with respect to their own synchronised reference frames, while they travel in a sufficiently small neighbourhood of the selected incomplete causal geodesic. Correspondingly, our approach does not require the use of any additional artificial structure such as any of the boundary constructions, which, in particular, ensures that our construction is free of the defects of the b -boundary construction.

This paper is structured according to the main steps on the course of our constructive proof, justifying the existence of the desired global extension, which are as follows. In section 2, some of the basic notions and results are recalled in connection with the Gaussian (resp., Gaussian null) coordinate systems. Then, in section 3, we select a sufficiently small neighbourhood \mathcal{U} of a final segment of an incomplete non-extendible timelike (resp., null) geodesic γ so that Gaussian (resp., Gaussian null) coordinates can be defined everywhere on \mathcal{U} by making use of an $n - 1$ -parameter family of timelike (resp., null) geodesics, denoted by Γ . In section 4, the extendibility of the metric structure defined on \mathcal{U} is studied. Then, an intermediate extension is given by constructing an isometry map $\phi : (\mathcal{U}, g_{ab}|_{\mathcal{U}}) \rightarrow (\mathcal{U}^*, g_{ab}^*)$ which provides an extension of $(\mathcal{U}, g_{ab}|_{\mathcal{U}})$ so that for those causal geodesics $\bar{\gamma} \in \Gamma$ which are sufficiently close to γ and are also incomplete and inextendible in (M, g_{ab}) the causal geodesics $\Phi \circ \bar{\gamma}$ can be extended in $(\mathcal{U}^*, g_{ab}^*)$. Once we have this intermediate extension, in section 5, a characterisation of spacetimes possessing topological singularities is given. Finally, with the help of the intermediate extension $\phi : (\mathcal{U}, g_{ab}|_{\mathcal{U}}) \rightarrow (\mathcal{U}^*, g_{ab}^*)$, the desired global extension, $\Phi : (M, g_{ab}) \rightarrow (\widehat{M}, \widehat{g}_{ab})$, will be constructed in section 6. The paper is closed by our final remarks and by addressing some of the open issues.

2 Gaussian and Gaussian null coordinate systems

In each particular cases, whenever a global extension of a spacetime could be performed it was always done by introducing suitable new coordinates. This section is to give a brief account on the types of local coordinate systems we shall apply in constructing the desired global extensions of causal geodesically incomplete spacetimes.

Let $\gamma : (t_1, t_2) \rightarrow M$ be a future directed and future incomplete⁵ timelike (resp., null) geodesic curve. Assume that t is an affine parameter along γ , and denote by v^a the associated tangent vector field. It is well-known that Gaussian (resp., Gaussian null) coordinates can be defined [40] (see also [42]), in a sufficiently small neighbourhood of any point $p = \gamma(t_0)$ of γ , where $t_0 \in (t_1, t_2)$, as follows:

Suppose first that γ is a timelike geodesic. Then, without loss of generality, we shall assume that t is the proper time along γ . Let Σ be a smooth spacelike hypersurface meeting γ orthogonally at $p = \gamma(t_0)$, and let (x^1, \dots, x^{n-1}) be coordinates on Σ . Choose then v^a

⁵Hereafter, we always assume that the incomplete causal curves are future directed and future incomplete. Note, however, that all the following constructions and results, presented for the case of future incomplete causal geodesics, can be recast to be applicable for past incomplete geodesics, as well, by replacing the words ‘future’ and ‘past’ everywhere systematically.

to be the smooth (future directed) unit norm, $g_{ab}v^av^b = -1$, timelike vector field which is everywhere normal to Σ . This vector field is, in fact, a unique smooth extension of the tangent of γ at $p = \gamma(t_0)$ to Σ . Consider now the $n - 1$ -parameter congruence of timelike geodesics, Γ , starting at the points of Σ with tangent v^a . Since Σ and v^a are smooth these geodesics do not intersect in a sufficiently small neighbourhood V of Σ . Extend now the functions x^1, \dots, x^{n-1} to V by keeping their values to be constant along the members of Γ . Then, by choosing as our n^{th} coordinate function x^n on V the proper time, t , along the members of Γ , that is synchronised so that $x^n = t_0$ on Σ , the functions x^1, \dots, x^n give rise to local coordinates on V . The yielded coordinate system is called to be *Gaussian*. In these coordinates the spacetime metric, g_{ab} , can be seen to take the form

$$ds^2 = -dt^2 + g_{\alpha\beta} dx^\alpha dx^\beta, \quad (2.1)$$

where $g_{\alpha\beta}$ is a $(n - 1) \times (n - 1)$ positive definite matrix the components of which are smooth functions of the coordinates (x^1, \dots, x^n) , and the Greek indices take the values $1, 2, \dots, n - 1$.

If γ happens to be a null geodesic the construction is different, nevertheless, an $n - 1$ -parameter congruence of null geodesics and Gaussian null coordinates can also be defined in a sufficiently small neighbourhood of any point $p = \gamma(t_0)$ as follows: Let Λ be a smooth $n - 2$ -dimensional spacelike surface orthogonal to γ at $p = \gamma(t_0)$. Choose v^a to be a smooth extension of the tangent of γ at $p = \gamma(t_0)$ to Λ so that v^a is normal to and null on Λ . Consider now the unique future directed smooth null vector field u^a on Λ , which is orthogonal to Λ and is normalised so that $g_{ab}u^au^b = -1$ throughout Λ . Denote by $\tilde{\Gamma}$ and $\hat{\Gamma}$ the $n - 2$ -parameter families of null geodesics the members of which start on Λ with tangent v^a and u^a , respectively. Moreover, denote by t and r the associated affine parameters along the null geodesics generating $\tilde{\Gamma}$ and $\hat{\Gamma}$ which are supposed to be synchronised so that $t = t_0$ and $r = 0$ on Λ . Since Λ and u^a are smooth there exists, in the null hypersurface spanned by the members of $\hat{\Gamma}$, a sufficiently small neighbourhood, Σ , of Λ so that the geodesics belonging to $\hat{\Gamma}$ do not intersect within Σ . By Lie propagating Λ within this neighbourhood we get a one-parameter family of $n - 2$ -dimensional spacelike surfaces Λ_r . Denote by $\tilde{\Gamma}_r$ the associated $n - 2$ -parameter family of null geodesics which meet Λ_r orthogonally and which are also transversal to Σ . For each value of r there exist a sufficiently small neighbourhood \tilde{V}_r of Λ_r such that the members of $\tilde{\Gamma}_r$ do not intersect within \tilde{V}_r . Denote by Γ the $n - 1$ -parameter family of null geodesics consisting of the members of the congruences $\tilde{\Gamma}_r$. Then the union of \tilde{V}_r give rise to a neighbourhood V of Σ within which *Gaussian null* coordinates can be defined as follows: Let (x^1, \dots, x^{n-2}) be arbitrary coordinates on Λ and extend them to Σ by keeping them constant along the members of $\hat{\Gamma}$. They, along with $x^{n-1} = r$, give rise to coordinates (x^1, \dots, x^{n-1}) on Σ . Let v^a be the unique smooth extension of the null vector field u^a from Λ to Σ determined so that v^a is tangent to the members of $\tilde{\Gamma}_r$ everywhere on Σ and is normalised so that $g_{ab}v^av^b = -1$, with $u^a = (\partial/\partial r)^a$, on Σ . Denote by t the associated affine parameter along the members of Γ which is synchronised so that $t = t_0$ at the points of Σ . Extend the coordinates (x^1, \dots, x^{n-1}) from Σ to V by keeping them to be constant along the members of Γ , while choose as our n^{th} -coordinate function $x^n = t$ in V . These coordinates are called to be *Gaussian null coordinates* in which coordinates the

spacetime metric, g_{ab} , takes the form

$$ds^2 = g_{rr} dr^2 - 2 dr dt + 2 g_{rA} dr dx^A + g_{AB} dx^A dx^B, \quad (2.2)$$

where g_{rr} , g_{rA} , g_{AB} are smooth functions of the coordinates (x^1, \dots, x^n) , and g_{AB} is an $(n-2) \times (n-2)$ positive definite matrix, and the uppercase Latin indices take the values $1, 2, \dots, n-2$. It is worth emphasising that the above construction does also guarantee that the components g_{rr} and g_{rA} vanish on Σ . For a justification of the last assertion see, e.g., Appendix A of Ref. [20].

3 The selection of \mathcal{U}

This section is to find a subset \mathcal{U} of M suitable for the construction of the desired intermediate extension $\phi : (\mathcal{U}, g_{ab}|_{\mathcal{U}}) \rightarrow (\mathcal{U}^*, g_{ab}^*)$. This will be done by demonstrating first that whenever the tidal force component of the curvature tensor are bounded, with respect to a synchronised basis field, along the members of an $n-1$ -parameter family of timelike (resp., null) geodesics Γ then for a suitable choice of t_0 Gaussian (resp., Gaussian null) coordinates can be defined which are appropriate at least locally everywhere in subset of M containing final segments of the selected future incomplete timelike (resp., null) geodesics in Γ .

Before proceeding, in order to have a ‘quasi-locally’ well-defined reference system with respect to which the change of various quantities can be expressed properly, we introduce the notion of synchronised Gaussian (resp., Gaussian null) coordinate systems, as well as, synchronised orthonormal (resp., pseudo-orthonormal) basis fields along the members of the $n-1$ -parameter congruences of timelike (resp., null) geodesics. To get these synchronised systems we need to start with a more special choice for the base manifold Σ (resp., Λ) of Gaussian (resp., Gaussian null) coordinate systems than the one applied in the previous section. More concretely, start by choosing Q to be a sufficiently small open neighbourhood of the origin in the linear subspace $T_{p=\gamma(t_0)}^\perp(M)$ of the $n-1$ (resp., $n-2$) -dimensional subspace of spacelike vectors orthogonal to v^a (resp., to v^a and u^a , where u^a is an arbitrarily chosen future directed null vector at p scaled so that $v_a u^a = -1$). Chose then Σ (resp., Λ) to be the image of Q under the action of the exponential map⁶, i.e., Σ (resp., Λ) = $\exp[Q]$. Accordingly, Σ (resp., Λ) is generated by spacelike geodesics starting at $p = \gamma(t_0)$ with tangent orthogonal to v^a (resp., in the null case orthogonal to both v^a and u^a) at p . The vector $v^a \in T_p(M)$ (resp., in the null case, as well as, $u^a \in T_p(M)$) can then be extended first to Σ (resp., Λ) by parallelly propagating it (resp., them) along the spacelike geodesics generating Σ (resp., Λ). In the timelike case the congruence Γ and the associated Gaussian coordinate system on V get immediately to be uniquely determined. In the null case the future directed vector field u^a on Λ uniquely determines the congruence $\hat{\Gamma}$, which spans the null hypersurface Σ through Λ . The synchronised affine parametrisation of the members of

⁶Recall that the exponential map $\exp : T_p(M) \rightarrow M$ is defined by taking a vector $X^a \in T_p(M)$ and proceeding along the geodesic from p in the direction of X^a a unit distance as measured in the affine parameter determined by X^a . The exponential map is known to be a local diffeomorphism between a sufficiently small open neighbourhood of the origin in $T_p(M)$ and that of $p \in M$ (see e.g. [29, 40]).

$\widehat{\Gamma}$ immediately determines then both the foliation of Σ by the $n - 2$ -dimensional spacelike surfaces Λ_r , as well as, the future directed vector field u^a on Σ . A future directed smooth vector field v^a on Σ , along with the associated null congruence Γ and the Gaussian null coordinate system on V , gets to be uniquely determined by requiring v^a to be orthogonal to the $n - 2$ -dimensional spacelike surfaces Λ_r throughout Σ , and also by scaling it so that $v_a u^a = -1$ on Σ .

A synchronised orthonormal (resp., pseudo-orthonormal) basis field $\{e_{(\mathbf{a})}^a\}$, along the members of the associated $n - 1$ -parameter congruence of timelike (resp., null) geodesics Γ can now be defined as follows. Start with an orthonormal (resp., pseudo-orthonormal) basis $\{e_{(\mathbf{a})}^a\} \subset T_p(M)$, here the name index \mathbf{a} takes the values $1, 2, \dots, \mathbf{n}$ which is chosen so that $e_{(\mathbf{n})}^a = v^a$ (resp., $e_{(\mathbf{n})}^a = v^a$ and $e_{(\mathbf{n}-1)}^a = u^a$) at p . Extend then $\{e_{(\mathbf{a})}^a\}$ from p by parallelly propagating it first along the spacelike geodesics generating Σ (resp., Λ). In the null case v^a and u^a are already defined as smooth null vector fields on Σ , which are also scaled there so that $v_a u^a = -1$, thereby we extend the pseudo-orthonormal basis field $\{e_{(\mathbf{a})}^a\}$ from Λ to Σ by requiring $e_{(\mathbf{n})}^a = v^a$ and $e_{(\mathbf{n}-1)}^a = u^a$ throughout Σ . To get the desired pseudo-orthonormal basis field $\{e_{(\mathbf{a})}^a\}$ on Σ , in addition, suitable spacelike vector fields $\{e_{(1)}^a \dots e_{(\mathbf{n}-2)}^a\}$ need also to be defined there. This can be done as follows. Take first the parallel transport of the spacelike vector fields $e_{(1)}^a, \dots, e_{(\mathbf{n}-2)}^a$ from Λ —they have already been defined there—along the members of $\widehat{\Gamma}$ to Σ . The yielded spacelike vector fields will be denoted by $e'_{(1)}^a, \dots, e'_{(\mathbf{n}-2)}^a$. By making use of v^a , u^a and these parallelly propagated fields define now the smooth spacelike vector fields $e_{(1)}^a, \dots, e_{(\mathbf{n}-2)}^a$ on Σ as

$$e_{(\mathbf{a})}^a = e'_{(\mathbf{a})}^a + (g_{ef} v^e e'^f_{(\mathbf{a})}) \cdot u^a, \quad (3.3)$$

where \mathbf{a} takes the values $1, 2, \dots, \mathbf{n} - 2$. These spacelike unite vector fields, by construction, are orthogonal to both v^a and u^a on Σ , and also they are pairwise orthogonal to each other so they together with $e_{(\mathbf{n})}^a = v^a$ and $e_{(\mathbf{n}-1)}^a = u^a$ comprise the desired basis field $\{e_{(1)}^a \dots e_{(\mathbf{n})}^a\}$ on Σ . Since the spacelike unite vector fields $e_{(\mathbf{a})}^a$ are orthogonal to both v^a and u^a on Σ they can also be seen to be tangent to the $n - 2$ -dimensional spacelike surfaces Λ_r .

Finally, in both, the timelike and the null, cases we extend $\{e_{(\mathbf{a})}^a\}$ —say to the neighbourhood V of Σ —by parallelly propagating the basis field $\{e_{(\mathbf{a})}^a\}$ from Σ along the members of the timelike (resp., null) geodesic congruence Γ . Clearly, the construction guarantees that the relation $e_{(\mathbf{n})}^a = v^a$ will hold everywhere along the members of Γ . Notice also that in the null case the vector field $e_{(\mathbf{n}-1)}^a$, in general, need not to be orthogonal to the $n - 2$ -dimensional spacelike surfaces $\Lambda_{t=\text{const}, r=\text{const}}$ apart from Σ . Hereafter, we shall always use the above type of Gaussian (resp., Gaussian null) coordinate systems, timelike (resp., null) geodesic congruence Γ and orthonormal (resp., pseudo-orthonormal) basis fields which all will be referred as being *synchronised*.

Notice that in spite of the fact that the Gaussian or Gaussian null coordinate systems may only be defined in a sufficiently small neighbourhood V of Σ the frame field $\{e_{(\mathbf{a})}^a\}$ may always be defined everywhere along any individual member $\bar{\gamma}$ of the causal geodesic congruence Γ

simply by parallel transporting $\{e_{(a)}^a\} \in T_{\bar{\gamma} \cap \Sigma}(M)$ along $\bar{\gamma}$. Similarly, by keeping the values of the coordinates of the intersection $\bar{\gamma} \cap \Sigma$ of $\bar{\gamma} \in \Gamma$ and Σ to be constant along the members of the congruence Γ to each point $q = \bar{\gamma}(t_q)$ an $(x^1, \dots, x^n) \in \mathbb{R}^n$ with $x^n = t_q$ can be assigned uniquely. Consider now the subset \mathcal{U} of \mathbb{R}^n defined as follows: $(x^1, \dots, x^n) \in \mathbb{R}^n$ belongs to \mathcal{U} if there exists $\bar{\gamma} \in \Gamma$ so that $x^n \in \text{dom}(\bar{\gamma})$ ⁷ and (x^1, \dots, x^{n-1}) are the coordinates of $\bar{\gamma} \cap \Sigma$ on Σ . We define now the map $\psi : \mathcal{U} \rightarrow M$ by requiring $\psi(x^1, \dots, x^n)$ to be the point $\bar{\gamma}(x^n) \in M$ along the geodesic $\bar{\gamma} \in \Gamma$ so that x^1, \dots, x^{n-1} are the coordinates of $\bar{\gamma} \cap \Sigma$ on Σ . Clearly, then all of the spacetime points covered by the members of Γ are represented in $\psi[\mathcal{U}]$. Moreover, although, by the above construction, ψ is smooth, in general, it is not one-to-one on the entire of \mathcal{U} . Thereby, usually V is only a proper subset of $\psi[\mathcal{U}]$.

Utilising now the above defined basis fields the tidal force tensor components of the Riemann tensor, along the members of Γ and with respect to a synchronised orthonormal (resp., pseudo-orthonormal) basis field $\{e_{(a)}^a\}$, can be defined as

$$R_{anbn} = R_{abcd}e_{(a)}^a v^b e_{(b)}^c v^d, \quad (3.4)$$

where the indices a, b take the values $1, 2, \dots, n-1$ (resp., $1, 2, \dots, n-2$).

In characterising the behaviour of the geodesics belonging to Γ the timelike (resp., null) sectional curvature function and the second fundamental form of Σ (resp., Λ) with respect to these vorticity free congruences do also play important role. The timelike (resp., null) sectional curvature function, along a timelike (resp., null) geodesic γ , is defined in terms of the timelike (resp., null) sectional curvature $K(\gamma, Z^a)$ (resp., $K_v(\gamma, Z^a)$). The latter is always defined with respect to two-dimensional timelike (resp., null) linear subspaces in $T_{\gamma(t)}M$, generated by the tangent v^a of γ and a spacelike vector field Z^a , in the null case Z^a is also assumed to be orthogonal to v^a , and they are given as [2, 28] (see also [42])

$$K(\gamma, Z^a) = \frac{R_{abcd}Z^a v^b Z^c v^d}{(v^e v_e)(Z^f Z_f) - (v^e Z_e)^2} \quad \left(\text{resp. } K_v(\gamma, Z^a) = \frac{R_{abcd}Z^a v^b Z^c v^d}{Z^e Z_e} \right). \quad (3.5)$$

The timelike (resp., null) sectional curvature function is defined then as the infimum of the timelike (resp., null) sectional curvatures $K(\gamma, Z^a)$ (resp., $K_v(\gamma, Z^a)$), along a timelike (resp., null) geodesic γ , i.e.,

$$K(t) = \inf\{K(\gamma, Z^a) \mid Z^a \in T_{\gamma(t)}M \text{ is spacelike}\} \quad (3.6)$$

$$\left(\text{resp., } K(t) = \inf\{K_v(\gamma, Z^a) \mid Z^a \in T_{\gamma(t)}M \text{ is spacelike and orthogonal to } v^a \} \right). \quad (3.7)$$

Let us assume, as above, that we have a synchronised family of timelike (resp., null) geodesic curves Γ , and also that a synchronised orthonormal (resp., pseudo-orthonormal) basis field $\{e_{(a)}^a\}$ is chosen along the members of Γ . Then the second fundamental form χ_{ab} of Σ (resp., Λ), with respect to Γ and $\{e_{(a)}^a\}$, may be defined as

$$\chi_{ab} = P_a^e P_b^f \nabla_{(e} v_{f)}, \quad (3.8)$$

⁷Here $\text{dom}(\bar{\gamma})$ denotes the domain (\bar{t}_1, \bar{t}_2) of a causal geodesic $\bar{\gamma} : (\bar{t}_1, \bar{t}_2) \rightarrow M$.

where the operator $P_a^b = \delta_a^b + v_a v^b$ (resp., $P_a^b = \delta_a^b + v_a u^b + u_a v^b$) projects at each point $p \in M$ the tangent space $T_p(M)$ into the $n-1$ (resp., $n-2$) -dimensional subspace $T_p^\perp(M)$ orthogonal to v^a (resp., in the null case to both v^a and u^a). The norm $\|\chi\|_p$ of χ_{ab} at a point $p \in M$ is defined then as

$$\|\chi\|_p = \sup_{|Y^a|_p=1} \left\{ \left| h^{ae} \chi_{ef} Y^f \right|_p \right\}, \quad (3.9)$$

where h^{ab} denotes the inverse of the induced metric $h_{ab} = P_a^e P_b^f g_{ef}$ on $T_p^\perp(M)$, and $|Z^a|_p$ denotes, for any spacelike vector $Z^a \in T_p^\perp(M)$, the norm $|Z^a|_p = (h_{ef} Z^e Z^f)^{1/2}$ determined by the induced metric h_{ab} .

In returning to the problem of selecting a suitable subset \mathcal{U} of M assume, for the moment being, that we have made a specific choice for $t_0 \in (t_1, t_2)$ and for Σ whence the $(n-1)$ -parameter family of timelike (resp., null) geodesics Γ has been fixed. Suppose also that the tidal force tensor components R_{anbn} are bounded, with respect to a synchronised orthonormal (resp., pseudo-orthonormal) basis fields, along the members of Γ . Then, it follows from Corollaries 3.2.2 and 3.2.3 of [42], and also from the remarks following them, that there must exist a positive real number K_0 so that $-K_0$ is a uniform lower bound to the timelike (resp., null) sectional curvature functions $K_{\bar{\gamma}}(t)$ along the members $\bar{\gamma} \in \Gamma$.⁸ Then, in virtue of Corollary 3.2.4 of [42] neither of the non-trivial Jacobi fields may vanish along any member $\bar{\gamma}$ of Γ within the affine parameter interval $(t_0, t_0 + \arctan(\sqrt{K_0}/\|\chi_{\bar{\gamma} \cap \Sigma}\|)/\sqrt{K_0})$, where $\|\chi_{\bar{\gamma} \cap \Sigma}\|$ denotes the norm of the second fundamental form at the intersection of $\bar{\gamma}$ and Σ . Recall also that according to the choice we have made for Σ (resp., Λ) in defining the synchronised Gaussian (resp., Gaussian null) coordinate systems guarantees that the second fundamental form χ_{ab} of Σ (resp., Λ) vanishes at $p = \gamma(t_0)$. Therefore, by choosing a sufficiently small open neighbourhood⁹ σ_{t_0} of p in Σ , in virtue of the vanishing of the second fundamental form χ_{ab} at p , it can be ensured that for its norm $\|\chi\| < \varepsilon_{[\chi, \sigma_{t_0}]}$ holds everywhere on σ_{t_0} , where $\varepsilon_{[\chi, \sigma_{t_0}]}$ is a suitably small fixed positive number. Assume now that such a neighbourhood σ_{t_0} has been chosen. Notice that the associated selection rule does also ensure that σ_{t_0} can be chosen so that it possesses compact closure in Σ .

It follows then that there cannot occur a point which would be conjugate to σ_{t_0} along any member $\bar{\gamma}$ of Γ within the affine parameter interval $(t_0, t_0 + \arctan(\sqrt{K_0}/\varepsilon_{[\chi, \sigma_{t_0}]})/\sqrt{K_0})$. Then, by making use of the fact that the closure of σ_{t_0} is compact in Σ , we may choose σ_{t_0}

⁸ Here and in the following whenever the phrase “uniformly bounded along the members of a congruence of causal geodesic curves, Γ ,” appears it is always meant that the corresponding quantities are bounded along the individual members of Γ , and also that these bounds have an upper bound.

⁹ A neighbourhood σ_{t_0} of $p = \gamma(t_0)$ of the desired type may be defined as follows. Whenever γ is timelike σ_{t_0} is chosen to be the image $\sigma_{t_0} = \exp[Q']$ of a sufficiently small $n-1$ -dimensional neighbourhood $Q' \subset Q$ of the origin in $T_p^\perp(M)$ under the action of the exponential map. If γ is null—i.e., whenever the $n-2$ -dimensional $Q' \subset Q$ is so that $\exp[Q'] \subset \Lambda$ —define the open neighbourhood σ_{t_0} of p in Σ to be the Lie transport of $\exp[Q']$ along the null geodesics with tangent u^a spanning Σ , i.e., $\sigma_{t_0} = \{\varphi_r[\exp[Q']] \mid r \in (-\tau, \tau)\}$, where φ_r denotes the local *one*-parameter family of diffeomorphisms induced by the vector field u^a on Σ , and the affine parameter r takes values from the interval $(-\tau, \tau)$ for some sufficiently small positive number τ .

to be sufficiently small so that, for some small positive number ϵ , the inequality

$$t_0 + \arctan(\sqrt{K_0}/\varepsilon_{[\chi, \sigma_{t_0}]})/\sqrt{K_0} > t_2 + \epsilon \quad (3.10)$$

also holds. If this happens the above choice we have made for $t_0 \in (t_1, t_2)$, Σ , Γ and σ_{t_0} is already suiting to the selection process, described below, which by the end of this section singles out the desired subset $\mathcal{U} \subset M$.

If (3.10) cannot be guaranteed to hold for the particular choice that has been made above we may proceed as follows.

Definition 3.1 *Consider all the possible sequences $\{[t_{0(i)}, \sigma_{t_{0(i)}}]\}$ chosen so that the real numbers $t_{0(i)} \in (t_1, t_2)$ converge to t_2 as i tends to infinity, while $\sigma_{t_{0(i)}}$ is chosen to be a sufficiently small subset of the timelike (resp., null) hypersurface $\Sigma_{t_{0(i)}}$. For any particular choice of a member $[t_{0(i)}, \sigma_{t_{0(i)}}]$ of such a sequence $\{[t_{0(i)}, \sigma_{t_{0(i)}}]\}$ we may consider the tidal force tensor components R_{anbn} of the curvature tensor, with respect to a synchronised orthonormal (resp., pseudo-orthonormal) basis field $\{e_{(a)}^a\}$, along the members of the causal geodesic congruence $\Gamma_{[t_{0(i)}, \sigma_{t_{0(i)}}]}$. Then the following two complementary cases may occur: Either there exists a sequence $\{[t_{0(i)}, \sigma_{t_{0(i)}}]\}$ and a positive real number that is an upper bound to the components of the tidal force tensor along the members of $\Gamma_{[t_{0(i)}, \sigma_{t_{0(i)}}]}$ for arbitrary choices of the members $[t_{0(i)}, \sigma_{t_{0(i)}}]$ of the sequence $\{[t_{0(i)}, \sigma_{t_{0(i)}}]\}$, or such a sequence and an associated upper bound do not exist. In the later case we shall say that γ does terminate on a tidal force tensor singularity.*

As an immediate example for a spacetime possessing a tidal force singularity see Example 6.2 below. It is worth keeping in mind, what is also clearly demonstrated by this example, that the above introduced notion is quasi-local in the sense that γ may terminate on a “tidal force tensor singularity” meanwhile the components of the tidal force tensor remain completely regular along a final segment of γ . Nevertheless, it is always true that either of the components of the tidal force tensor necessarily blows up along some of the arbitrarily close causal geodesic curves. If this happens the existence of a global extension, so that the image of γ could also be extended, is excluded.

Assume now that γ does not terminate on a tidal force tensor singularity, i.e., for some sequence $\{[t_{0(i)}, \sigma_{t_{0(i)}}]\}$ there exists an upper bound, which simultaneously bounds the components of the tidal force tensor along the members of the causal geodesic congruences $\{\Gamma_{[t_{0(i)}, \sigma_{t_{0(i)}}]}\}$. Then, there also has to exist a positive real number K_0^* so that $-K_0^*$ is a universal lower bound to the timelike (resp., null) sectional curvature functions along the members of the associated congruences. This, in particular, implies then that by choosing t_0 sufficiently close to t_2 , and also by choosing σ_{t_0} so that $\varepsilon_{[\chi, \sigma_{t_0}]}$ is sufficiently small it can be guaranteed that the inequality

$$t_0 + \arctan(\sqrt{K_0^*}/\varepsilon_{[\chi, \sigma_{t_0}]})/\sqrt{K_0^*} > t_2 + \epsilon \quad (3.11)$$

will hold for some small positive number ϵ . In what follows we shall assume that γ does not terminate on a “tidal force tensor singularity”, and also that an appropriate choice for $t_0 \in (t_1, t_2)$, σ_{t_0} and ε has been made.

Consider, now, the subset $\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}$ of $\mathcal{U} \subset \mathbb{R}^n$ defined so that

$$\mathcal{U}_{[\sigma_{t_0}, \varepsilon]} := \{(x^1, \dots, x^n) \in \mathcal{U} \mid \psi(x^1, \dots, x^{n-1}, t_0) \in \sigma_{t_0} \text{ and } x^n \in [t_0, t_2 + \varepsilon)\}. \quad (3.12)$$

In proceeding, select first an arbitrary member $\bar{\gamma}$ of Γ starting at a point of σ_{t_0} . Then, by applying the argument of the proof of Proposition 3.2.5 of [42] to $\bar{\gamma}$, it can be justified that to any point q of $\bar{\gamma}$ there must exist a sufficiently small open neighbourhood \mathcal{O}_q so that the Gaussian (resp., Gaussian null) coordinate functions get to be locally well-defined coordinates on \mathcal{O}_q . Notice that \mathcal{O}_q need not to be a subset of V where the Gaussian (resp., Gaussian null) coordinates are, by construction, well-defined. Then, by applying this argument of Proposition 3.2.5 of [42] to the individual members of the synchronised causal geodesic congruence $\Gamma_{[t_0, \sigma_{t_0}]}$ simultaneously, it follows that for a suitable choice of ε the Gaussian (resp., Gaussian null) coordinate functions get to be, at least, locally well-defined coordinates on $\psi[\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}]$. More precisely, by making use of the above outlined argument the following proposition can be seen to hold.

Proposition 3.1 *Let $\gamma : (t_1, t_2) \rightarrow M$ be a future incomplete timelike (resp., null) geodesic curve which does not terminate on a tidal force tensor singularity. Then, $t_0 \in (t_1, t_2)$, σ_{t_0} , a small positive number ε —the value of which depends on that of $\varepsilon_{[\chi, \sigma_{t_0}]}$ and on the uniform lower bound, $-K_0^*$, of the timelike (resp., null) sectional curvature functions along the members of $\Gamma_{[t_0, \sigma_{t_0}]}$ ¹⁰—, and also a subset $\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}$ of $\mathcal{U} \subset \mathbb{R}^n$ can be chosen so that to any point $q \in \text{int}(\psi[\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}]) \subset M$ there exists a sufficiently small open neighbourhood $\mathcal{O}_q \subset \mathcal{U}_{[\sigma_{t_0}, \varepsilon]}$ of a preimage $x^\alpha(q) \in \mathbb{R}^n$ of q such that the restriction $\psi|_{\mathcal{O}_q}$ of ψ is a diffeomorphism between \mathcal{O}_q and its image $\mathcal{O}_q = \psi[\mathcal{O}_q]$.*

It is guaranteed by the above construction that for arbitrary choice of $q \in \text{int}(\psi[\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}])$ there exists (at least one) $\mathcal{O}_q \subset M$ neighbourhood such that the Gaussian (resp., Gaussian null) coordinate functions—introduced in Sections 2 and 3—determine well-defined coordinates on $\mathcal{O}_q = \psi[\mathcal{O}_q]$. Moreover, as the tidal force components of the curvature tensor are uniformly bounded along the members of Γ not only the existence of a uniform lower bound on timelike (resp., null) sectional curvature function along the members of Γ is guaranteed—this is what is really needed to prove Proposition 3.1—but a uniform upper bound to these quantities also exist. This later property is also of fundamental importance since without having it the appearance of certain “fountain type” behaviour—the type which occurs, e.g., in case of Killing orbits close to a bifurcation surface—of the geodesic congruence Γ while $t \rightarrow t_2$ along γ could not be excluded. It can be justified, e.g., by recalling Proposition 3.1 of [14], that neither the unbounded compression nor the unbounded expansion of Γ is tolerated by the conditions of the above proposition. Thereby, neither the close up nor the extreme opening up of the null cones may occur on $\psi[\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}]$. In fact, the assumptions we have applied in selecting $\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}$ guarantee that this type of behaviour is excluded on the entire of the closure of $\psi[\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}]$ which, in particular, implies that the components of the

¹⁰Henceforth, to simplify the applied notation we shall drop the indices of $\Gamma_{[t_0, \sigma_{t_0}]}$. This suits to the fact that the original congruence Γ —say by choosing Σ to be sufficiently small—may also be guaranteed to possess all the above discussed properties of $\Gamma_{[t_0, \sigma_{t_0}]}$.

metric, as they appear in the line element (2.1) (resp., (2.2)), are bounded on the subsets \mathcal{O}_q of $\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}$. Notice also that the pairs of the form $(\mathcal{O}_q, g_{ab}|_{\mathcal{O}_q})$ may be considered to be spacetimes on their own right whence, in particular, the causal relations make immediate sense on them.

Assume now that γ is timelike, and also that Γ is a synchronised $n - 1$ -parameter congruence of timelike geodesics, $q \in \text{int}(\psi[\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}])$, furthermore, that \mathcal{O}_q is a convex open neighbourhood of a preimage $x^\alpha(q) \in \mathbb{R}^n$ of q , with compact closure in $\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}$ so that the restriction $\psi|_{\mathcal{O}_q}$ of ψ to \mathcal{O}_q is a diffeomorphism between \mathcal{O}_q and $\mathcal{O}_q = \psi[\mathcal{O}_q]$. Then, it is guaranteed by the selection of \mathcal{O}_q that there must exist a subfamily Γ' of Γ so that the members of Γ' do not meet within \mathcal{O}_q . Moreover, since for each member of $\gamma' \in \Gamma'$ the intersection of γ' and σ_{t_0} is unique we also have that both the projection map $\pi : \mathcal{O}_q \rightarrow \sigma_{t_0}$, as well as, the pre-image $\varsigma_{t_0} = \psi^{-1}[\sigma_{t_0}] \subset \mathbb{R}^{n-1} \times \{t_0\}$ of σ_{t_0} by ψ^{-1} are well-defined. In addition, the projection of \mathcal{O}_q to σ_{t_0} , i.e., the set $(\psi^{-1} \circ \pi)[\mathcal{O}_q]$, may also be guaranteed to be a convex subset of ς_{t_0} .

Choose, now, an arbitrary point $s \in \partial J^-(q) \cap \mathcal{O}_q$, and let $\tilde{\lambda}$ be a curve lying on $\partial J^-(q) \cap \mathcal{O}_q$ connecting q and s defined as follows. Consider first the straight line segment connecting $x^\alpha(\pi(q))$ and $x^\alpha(\pi(s))$ in ς_{t_0} , and denote by λ the image of this straight line segment by the map ψ which is a curve in σ_{t_0} . Since $\partial J^-(q) \cap \mathcal{O}_q$ is an achronal subset of \mathcal{O}_q the restriction $\hat{\pi} = \pi|_{\partial J^-(q) \cap \mathcal{O}_q}$ of the projection map $\pi : \mathcal{O}_q \rightarrow \sigma_{t_0}$ to $\partial J^-(q) \cap \mathcal{O}_q$ is one-to-one, thereby its inverse $\hat{\pi}^{-1} = (\pi|_{\partial J^-(q) \cap \mathcal{O}_q})^{-1}$ is well-defined. Denote by $\tilde{\lambda}$ the unique lift of λ by the map $\hat{\pi}^{-1}$ to $\partial J^-(q) \cap \mathcal{O}_q$. Notice that $\tilde{\lambda} = \hat{\pi}^{-1}[\lambda]$ is an achronal curve from q to s , and also that it has to be at least locally Lipschitz since it lies on the boundary $\partial J^-(q) \cap \mathcal{O}_q$ [29].

Now, we are ready to formulate the following lemma which, along with its counter part, see Lemma 3.2, will play an important role in justifying Proposition 4.1 below.

Lemma 3.1 *Suppose that γ is timelike, and assume the conditions of Proposition 3.1 hold. Denote by Γ the associated $n - 1$ -parameter congruence of synchronised timelike geodesics. Choose q to be an arbitrary point of $\text{int}(\psi[\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}])$, with a convex open neighbourhood $\mathcal{O}_q \subset \mathcal{U}_{[\sigma_{t_0}, \varepsilon]}$ of a preimage $x^\alpha(q) \in \mathbb{R}^n$ of q possessing compact closure in $\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}$, so that the restriction $\psi|_{\mathcal{O}_q}$ of ψ to \mathcal{O}_q is a diffeomorphism between \mathcal{O}_q and $\mathcal{O}_q = \psi[\mathcal{O}_q]$. Let $s \in \partial J^-(q) \cap \mathcal{O}_q$, and let $\tilde{\lambda}$ be the curve from q to s lying on $\partial J^-(q) \cap \mathcal{O}_q$ as defined above. Then, there exists a positive number $K > 0$ so that for the Euclidean length $L(\tilde{\lambda}; q, s)$ of $\tilde{\lambda}$ —with respect to the Gaussian coordinates (x^1, \dots, x^n) in $\mathcal{O}_q \subset \mathbb{R}^n$ —the inequality*

$$L(\tilde{\lambda}; q, s) \leq \sqrt{1 + K^2} \cdot \tilde{\rho}(\pi(q), \pi(s)) \quad (3.13)$$

holds, where $\tilde{\rho}(\pi(q), \pi(s))$ denotes the Euclidean distance of the projections $\pi(q)$ and $\pi(s)$ on σ_{t_0} .

Proof: Recall first that the line element of the Euclidean metrics $\tilde{\rho}$ and ρ , defined in terms of the Gaussian coordinates (x^1, \dots, x^n) in $\mathcal{O}_q \subset \mathbb{R}^n$, on σ_{t_0} and on $\mathcal{O}_q = \psi[\mathcal{O}_q]$ are given as $d\tilde{\rho}^2 = \sum_{\alpha=1}^{n-1} (dx^\alpha)^2$ and $d\rho^2 = d\tilde{\rho}^2 + dt^2$, respectively. It immediately follows then that

$|dx^\alpha| \leq d\tilde{\rho}$ is satisfied for each of the spatial coordinates with $\alpha = 1, 2, \dots, n-1$. Notice also that, since the null cones remain regular on the closure of $\psi[\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}]$,¹¹ the components of the metric are bounded on the subsets \mathcal{O}_q of $\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}$, i.e., there must exist a positive number $K > 0$ so that for each of the components $g_{\alpha\beta}$ in the line element (2.1) $|g_{\alpha\beta}| < K^2/(n-1)^2$ holds. These relations imply then that

$$0 \leq ds^2 = -dt^2 + g_{\alpha\beta} dx^\alpha dx^\beta \leq -dt^2 + K^2 d\tilde{\rho}^2 \quad (3.14)$$

is satisfied everywhere along the achronal curve $\tilde{\lambda}$. This, in particular, implies that the relation $dt^2 \leq K^2 d\tilde{\rho}^2$ holds along $\tilde{\lambda}$ in \mathcal{O}_q . Consequently, for the Euclidean length $L(\tilde{\lambda}; q, s)$ of $\tilde{\lambda}$, as it is measured with respect to the Gaussian coordinates (x^1, \dots, x^n) in $\mathcal{O}_q \subset \mathbb{R}^n$, we have

$$\begin{aligned} L(\tilde{\lambda}; q, s) &= \int_{\psi^{-1}[\tilde{\lambda}]} d\rho = \int_{\psi^{-1}[\tilde{\lambda}]} \sqrt{d\tilde{\rho}^2 + dt^2} = \int_{\lambda=\pi(\tilde{\lambda})} \sqrt{1 + \frac{dt^2}{d\tilde{\rho}^2}} \cdot d\tilde{\rho} \\ &\leq \sqrt{1 + K^2} \cdot \tilde{\rho}(\pi(q), \pi(s)), \end{aligned} \quad (3.15)$$

where the integration along the curve $\psi^{-1}[\tilde{\lambda}]$ can be made to be meaningful since $\tilde{\lambda}$ itself is guaranteed to be (at least) locally Lipschitz as it is lying on the achronal boundary of a past set $J^-(q) \cap \mathcal{O}_q$ (see, e.g., [29]) while the last integral along the smooth pre-images $\lambda = \pi(\tilde{\lambda})$ of $\tilde{\lambda}$ in ζ_{t_0} makes immediate sense. \square

Assume now that γ is null, and also that Γ is a synchronised $n-1$ -parameter congruence of null geodesics. Suppose, furthermore, that $(x^1, \dots, x^{n-2}, r, t)$ are synchronised Gaussian null coordinates defined on \mathcal{O}_q . Before providing the counterpart of Lemma 3.1 introduce the auxiliary coordinates $(\bar{x}^1, \dots, \bar{x}^{n-2}, \bar{r}, \bar{t})$ on \mathcal{O}_q by making use of the relations

$$\bar{t} = \frac{1}{\sqrt{2}}(t + r), \quad \bar{r} = \frac{1}{\sqrt{2}}(t - r) \quad \text{and} \quad \bar{x}^A = x^A. \quad (3.16)$$

Since the Gaussian null coordinates $(x^1, \dots, x^{n-2}, r, t)$ are well-defined on \mathcal{O}_q so they are the local coordinates $(\bar{x}^1, \dots, \bar{x}^{n-2}, \bar{r}, \bar{t})$. It is also straightforward to see that the line element of the metric g_{ab} , in the local coordinates $(\bar{x}^1, \dots, \bar{x}^{n-2}, \bar{r}, \bar{t})$, can be given as

$$ds^2 = \frac{1}{2} g_{rr} (d\bar{t} - d\bar{r})^2 - d\bar{t}^2 + d\bar{r}^2 + \sqrt{2} g_{rA} (d\bar{t} - d\bar{r}) d\bar{x}^A + g_{AB} d\bar{x}^A d\bar{x}^B. \quad (3.17)$$

Recall that the components g_{rr} and g_{rA} , with $A = 1, \dots, n-2$,—besides being bounded—vanish on Σ thereby the contribution of the corresponding terms can be ensured to be negligibly small, with respect to the others, by choosing t_0 to be sufficiently close to the upper bound t_2 of the affine parameter t . In addition, as $r = \frac{1}{\sqrt{2}}(\bar{t} - \bar{r})$ may always be chosen to be sufficiently small, there must exist a positive number $K' > 0$ so that the value of K' is independent of the choice of q and \mathcal{O}_q , and also that for arbitrary achronally related pairs of points the relation

$$0 \leq ds^2 \leq -d\bar{t}^2 + d\bar{r}^2 + K' \cdot g_{AB} d\bar{x}^A d\bar{x}^B \quad (3.18)$$

¹¹For its justification see the argument below Proposition 3.1.

holds.

We would like to indicate that the auxiliary coordinates $(\bar{x}^1, \dots, \bar{x}^{n-2}, \bar{r}, \bar{t})$ are introduced simply in order to overcome the following inconvenient situation. As opposed to the timelike case, to any $q \in \psi[\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}]$ the projections of the points of the null geodesic segment, $\gamma'_q = \gamma_q \cap [\partial J^-(q) \cap \mathcal{O}_q]$, where γ_q is the member of the synchronised $n - 1$ -parameter congruence of null geodesics Γ through q and the projection is defined with respect to Γ , coincide simply because γ'_q lies on $J^-(q) \cap \mathcal{O}_q$. However, as it will be shown below, since the coordinates $(\bar{x}^1, \dots, \bar{x}^{n-2}, \bar{r}, \bar{t})$ are well-defined on \mathcal{O}_q by making use of them a construction, analogous to the one that have been used in the timelike case, may also be applied here.

To start off consider first the \bar{t} -coordinate lines, the curves in \mathcal{O}_q along which all the other coordinates \bar{r} and \bar{x}^A are constant. Then, according to (3.17), the \bar{t} -coordinate lines comprise an $n - 1$ -parameter timelike congruence $\bar{\Gamma}$, in \mathcal{O}_q . Instead of the ‘screening’ hypersurface σ_{t_0} , that was applied in the proof of Lemma 3.1, in the present case it is more appropriate to use the space of \bar{t} -coordinate lines in \mathcal{O}_q which will be denoted by $\bar{\sigma}_q$. Choose, now, s to be an arbitrary point of $\partial J^-(q) \cap \mathcal{O}_q$, and let $\bar{\lambda}$ be a curve connecting q and s on the achronal set $\partial J^-(q) \cap \mathcal{O}_q$ defined as follows. Let q' and s' be arbitrary points of the timelike curves $\bar{\gamma}_q$ and $\bar{\gamma}_s$, which are the \bar{t} -coordinate lines through q and s , respectively. Consider, now, the straight line segment connecting the points $\bar{x}^\alpha(q')$ and $\bar{x}^\alpha(s')$ in $\bar{\mathcal{O}}_q \subset \mathbb{R}^n$, and denote by λ the image of this straight line segment by the map ψ . Since q and s are achronally related, and the congruence $\bar{\Gamma}$ is comprised by timelike curves there is precisely one member of $\bar{\Gamma}$ through each point of λ . Denote by $\bar{\Gamma}'$ the corresponding smooth one-parameter sub-congruence of $\bar{\Gamma}$. Again, since $\partial J^-(q) \cap \mathcal{O}_q$ is an achronal subset of \mathcal{O}_q each member of $\bar{\Gamma}'$ intersects $\partial J^-(q) \cap \mathcal{O}_q$ precisely once. Chose then $\bar{\lambda}$ to be the curve connecting q and s on $\partial J^-(q) \cap \mathcal{O}_q$ which is determined by the intersections of the members of the sub-congruence $\bar{\Gamma}'$ with $\partial J^-(q) \cap \mathcal{O}_q$. Clearly, then $\bar{\lambda}$, by construction, is an achronal curve from q to s , and it has to be at least locally Lipschitz.

The counterpart of Lemma 3.1 can now be formulated as follows.

Lemma 3.2 *Suppose now that γ is null, and assume the conditions of Proposition 3.1 hold. Denote by Γ the associated $n-1$ -parameter congruence of synchronised null geodesics. Choose q to be an arbitrary point of $\text{int}(\psi[\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}])$, with a convex open neighbourhood $\mathcal{O}_q \subset \mathcal{U}_{[\sigma_{t_0}, \varepsilon]}$ of a preimage $x^\alpha(q) \in \mathbb{R}^n$ of q possessing compact closure in $\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}$, so that the restriction $\psi|_{\mathcal{O}_q}$ of ψ to \mathcal{O}_q is a diffeomorphism. Let $\bar{\Gamma}$ be the $n - 1$ -parameter family of timelike curves in \mathcal{O}_q , $s \in \partial J^-(q) \cap \mathcal{O}_q$, and $\bar{\lambda}$ be the curve from q to s lying on $\partial J^-(q) \cap \mathcal{O}_q$ as defined above. Then, there exists a positive number $\bar{K} > 0$ so that for the Euclidean length $L(\bar{\lambda}; q, s)$ of $\bar{\lambda}$ —with respect to the Gaussian null coordinates $(x^1, \dots, x^{n-2}, r, t)$ in $\mathcal{O}_q \subset \mathbb{R}^n$ —, the inequality*

$$L(\bar{\lambda}; q, s) \leq \sqrt{1 + \bar{K}^2} \cdot \bar{\rho}(\bar{\gamma}_q, \bar{\gamma}_s) \quad (3.19)$$

holds, where $\bar{\rho}(\bar{\gamma}_q, \bar{\gamma}_s)$ denotes the Euclidean distance of the ‘points’ $\bar{\gamma}_q$ and $\bar{\gamma}_s$ in $\bar{\sigma}_q$.

Proof: Notice first that the line element of the metric determining the Euclidean distance

on the space of the \bar{t} -coordinate lines, $\bar{\sigma}_q$, reads as

$$d\bar{\rho}^2 = \sum_{A=1}^{n-2} (d\bar{x}^A)^2 + d\bar{r}^2. \quad (3.20)$$

This relation implies then that the inequalities $|d\bar{x}^A| \leq d\bar{\rho}$, for any $A = 1, \dots, n-2$, and also that $|d\bar{r}| \leq d\bar{\rho}$ hold. As above, since the null cones remain regular on the closure of $\psi[\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}]$ the components of the metric are (uniformly) bounded on the subsets \mathcal{O}_q of $\psi[\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}]$, i.e., there must exist a positive number $K > 0$ so that for each of the components g_{AB} in the line element (2.2) the inequalities $|g_{AB}| < K^2/(n-2)^2$ hold. These relations, along with (3.17) and (3.18), imply then that

$$\begin{aligned} 0 \leq ds^2 &= g_{rr}dr^2 - 2drdt + 2g_{rA}drdx^A + g_{AB}dx^A dx^B \\ &\leq -d\bar{t}^2 + d\bar{r}^2 + K' \cdot g_{AB} d\bar{x}^A d\bar{x}^B \leq -d\bar{t}^2 + \bar{K}^2 d\bar{\rho}^2, \end{aligned} \quad (3.21)$$

where $\bar{K}^2 = K' \cdot K^2$, holds everywhere along the achronal curve $\bar{\lambda}$. This, in particular, implies that the relation $d\bar{t}^2 \leq \bar{K}^2 d\bar{\rho}^2$ holds along $\bar{\lambda}$ in \mathcal{O}_q . Consequently, for the Euclidean length $L(\bar{\lambda}; q, s)$ of $\bar{\lambda}$, as it is measured with respect to the Gaussian null coordinates $(x^1, \dots, x^{n-2}, r, t)$ in $\mathcal{O}_q \subset \mathbb{R}^n$, we have that

$$\begin{aligned} L(\bar{\lambda}; q, s) &= \int_{\psi^{-1}[\bar{\lambda}]} d\rho = \int_{\psi^{-1}[\bar{\lambda}]} \sqrt{\sum_{A=1}^{n-2} (dx^A)^2 + dr^2 + dt^2} = \int_{\psi^{-1}[\bar{\lambda}]} \sqrt{\sum_{A=1}^{n-2} (d\bar{x}^A)^2 + d\bar{r}^2 + d\bar{t}^2} \\ &= \int_{\bar{\lambda}} \sqrt{1 + \frac{d\bar{t}^2}{d\bar{\rho}^2}} \cdot d\bar{\rho} \leq \sqrt{1 + \bar{K}^2} \cdot \bar{\rho}(\bar{\gamma}_q, \bar{\gamma}_s), \end{aligned} \quad (3.22)$$

holds, as we desired to show. \square

In the rest of this section we shall assume that the conditions of Proposition 3.1 are satisfied. Since our principal aim is to perform spacetime extensions based on the use of suitably chosen synchronised Gaussian (resp., Gaussian null) coordinate systems it is of obvious importance to know how far away from Σ these type of coordinates and the associated synchronised basis fields can be trustfully applied. In fact, what we really need to demonstrate—in order to find a subset \mathcal{U} of M suitable for the desired intermediate extension—is nothing else but that for appropriately chosen $t_0, \sigma_{t_0} \subset \Sigma$ and ε the map $\psi : \mathcal{U}_{[\sigma_{t_0}, \varepsilon]} \rightarrow M$ will be a one-to-one map. Clearly, then, ψ is much more than a simple local diffeomorphism it is, in fact, a true diffeomorphism between $\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}$ and its image $\psi[\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}]$. The associated desire is, however, immediately cooled down by the existence of those spacetime models that are yielded by “cutting and gluing regular spacetime regions” and in which, due to the applied arrangements, the existence of $\mathcal{U}_{[\sigma_{t_0}, \varepsilon]} \subset \mathbb{R}^n$ such that ψ would be one-to-one on $\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}$ is excluded (see, e.g., Refs. [11, 18] for relevant examples). To avoid the corresponding inconvenient situations we need to apply an assumption concerning the genericness of the underlying spacetime (M, g_{ab}) which is going to be manifested via the introduction of the following notion of topological singularity.

Definition 3.2 *An incomplete non-extendible timelike (resp., null) geodesic $\gamma : (t_1, t_2) \rightarrow M$ is said to terminate on a topological singularity if for any choice of t_0 , $\sigma_{t_0} \subset \Sigma$ and ε the set $\psi[\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}]$ is not simply connected. A spacetime itself is said to possess a topological singularity if it admits an incomplete non-extendible timelike (resp., null) geodesic curve which does terminate on a topological singularity.*

The above formulation of the concept of ‘topological singularity’, indicates that the existence of a singularity of this type is much more a topological rather than local physical or geometrical character of the underlying spacetime. Therefore, it is of obvious importance to know in what sort of circumstances a spacetime might admit a topological singularity. We shall return to this issue later, in section 5, after having our intermediate extension available for the case of spacetimes with non-topological singularities. It will be demonstrated there that to a globally hyperbolic spacetime which is otherwise regular but does possess a topological singularity it is always possible to construct a covering spacetime which is extendible and, more importantly, this covering spacetime has to be algebraically special. Thereby, the mere existence of a topological singularity implies that the associated covering spacetime cannot be generic, which, in turn, implies that the original spacetime cannot be generic either.

The next theorem is to justify the expectation that whenever $\gamma : (t_1, t_2) \rightarrow M$ does not terminate on a topological singularity for a suitable choice of $\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}$ the members of the $n - 1$ -parameter congruence of timelike (resp., null) geodesics do not intersect in $\psi[\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}]$, i.e., ψ is one-to-one everywhere on $\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}$.

Theorem 3.1 *Suppose that the assumptions of Proposition 3.1 hold, in particular, let $\gamma : (t_1, t_2) \rightarrow M$, t_0 , σ_{t_0} and $\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}$ be as they were specified there. Assume, in addition, that γ does not terminate on a topological singularity. Then $\psi : \mathcal{U}_{[\sigma_{t_0}, \varepsilon]} \rightarrow M$ is a one-to-one map between $\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}$ and its image $\psi[\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}]$.*

Proof: Hereafter, each of the proofs will be presented in details only for the timelike case. Nevertheless, whenever significant changes show up in the argument relevant for the null case they will also be spelled out at the end of each of the particular proofs.

Correspondingly, suppose now that γ is timelike and, on contrary to the above assertion, that the map $\psi : \mathcal{U}_{[\sigma_{t_0}, \varepsilon]} \rightarrow M$ is not one-to-one between $\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}$ and its image $\psi[\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}]$. Then there must exist $\gamma_1, \gamma_2 \in \Gamma$ starting on σ_{t_0} with tangent orthogonal to σ_{t_0} so that they intersect at certain point $q \in \psi[\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}] \subset M$.

Notice first that since γ_1 and γ_2 are not assumed to be infinitesimally close their intersection at q is not in an immediate conflict with the existence of a neighbourhood \mathcal{O}_q of $x^\alpha(q) \in \mathbb{R}^n$ so that the restriction $\psi|_{\mathcal{O}_q}$ of ψ to \mathcal{O}_q is a diffeomorphism between \mathcal{O}_q and $\mathcal{O}_q = \psi[\mathcal{O}_q]$. In the rest of the proof we shall show that, whenever our indirect hypotheses holds, there has to exist at least a one-parameter subfamily of Γ , containing γ_1 and γ_2 , so that all the members of this subfamily intersect at q which, in turn, leads to the conclusion that there exist points in σ_{t_0} that are conjugate to q along γ_1 and γ_2 . However,

the existence of conjugate points along the members of Γ is excluded by the assumptions of Proposition 3.1. Our assertion follows then from the fact that this conflict can only be resolved if the above indirect hypothesis is false.

In justifying the validity of the above assertion start by denoting the intersections of γ_1 and γ_2 with σ_{t_0} by p_1 and p_2 , respectively. Recall now that, although ψ may not be one-to-one on the entire of $\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}$ —this is, in fact, the very property which we would like to derive from our conditions—it is a local diffeomorphism from $\mathcal{U}_{[\sigma_{t_0}, \varepsilon]} \subset \mathbb{R}^n$ to $\psi[\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}] \subset M$. Chose, then, λ_q to be a closed curve through q within $\psi[\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}]$ which is composed by three pieces $\lambda_q = \lambda_{p_1 p_2} \cup \gamma_{p_1 q} \cup \gamma_{p_2 q}$, where $\lambda_{p_1 p_2}$ is a curve connecting p_1 and p_2 in σ_{t_0} , while $\gamma_{p_1 q}$ and $\gamma_{p_2 q}$ denote the segments of γ_1 and γ_2 from q to p_1 and to p_2 , respectively. Since $\gamma : (t_1, t_2) \rightarrow M$ does not terminate on a topological singularity $\psi[\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}]$ may be assumed to be simply connected, thereby, λ_q has to be homotopic to the trivial curve through q , i.e., there must exist a one-parameter family of curves λ_q^δ in $\psi[\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}]$, with $\delta \in [0, 1]$, so that $\lambda_q^{\delta=0} = q$ and $\lambda_q^{\delta=1} = \lambda_q$. For any fixed value of δ , denote by $\pi[\lambda_q^\delta]$ the projection of λ_q^δ by the congruence Γ , consisting of exactly those points of σ_{t_0} from which the points of λ_q^δ can be reached along a member of Γ . Notice that $\pi[\lambda_q^\delta]$ has to be a subset of the closure $\overline{\sigma_{t_0}}$ of σ_{t_0} , which itself was assumed to be a compact subset of Σ . Moreover, for each value of δ the set $\pi[\lambda_q^\delta]$ has to contain a curve $\hat{\lambda}_q^\delta$ which connects the points p_1 and p_2 in $\overline{\sigma_{t_0}}$. Since ψ is not guaranteed to be one-to-one yet it cannot be excluded that $\pi[\lambda_q^\delta]$ does also contain additional points not belonging to $\hat{\lambda}_q^\delta$. Nevertheless, since ψ itself is known to be local diffeomorphism, $\hat{\lambda}_q^\delta$ may be assumed to be at least a piecewise smooth a continuous curve for each fixed value of δ . Correspondingly, in virtue of Theorem 6.2.1 of [29] (see also [36]), the set of the limit points of the sequence of the curves $\{\hat{\lambda}_q^\delta\}$ in $\overline{\sigma_{t_0}}$ has to contain at least a continuous curve connecting the points p_1 and p_2 in $\overline{\sigma_{t_0}}$. This curve, which is a limit of the sequence $\{\hat{\lambda}_q^\delta\}$, will be denoted by $\hat{\lambda}_q^*$. Notice that by construction all the members of the congruence Γ starting at the points of $\hat{\lambda}_q^*$ do intersect at q . This, in particular, implies then that to any neighbourhood \mathcal{O}_{p_1} of p_1 in σ_{t_0} there exists $\bar{\gamma} \in \Gamma$ through q so that $\bar{\gamma}$ intersects \mathcal{O}_{p_1} , which, in turn, implies that p_1 has to be conjugate to q along γ_1 in $\psi[\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}] \subset M$. This, however, is impossible, since the conditions of Proposition 3.1 exclude the existence of this type of conjugate points.

The proof for the null case can be derived analogously starting with a self-evident replacement of the timelike geodesic congruence with null one, as well as, by taking into account that $\psi[\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}]$ is foliated then by null hypersurfaces spanned by $n - 2$ -parameter sub-congruences $\tilde{\Gamma}_r$, generated by null geodesics starting at the points of the $n - 2$ -dimensional spacelike surfaces Λ_r foliating σ_{t_0} . The most significant difference shows up in the following context. In the null case the conditions of Proposition 3.1 exclude apparently only the existence of conjugate points along a member $\bar{\gamma} \in \Gamma$ to the $n - 2$ -dimensional spacelike surface $\Lambda_{\bar{\gamma} \cap \sigma_{t_0}}$. Nevertheless, since in the null case whenever either of the name indices \mathbf{a}, \mathbf{b} takes the value $n - 1$ the tidal force tensor components $R_{\mathbf{a}\mathbf{b}\mathbf{n}}$ can be shown to be zero. Thereby the boundedness of the null sectional curvature function $K(t)$, for instance, along γ_1 is guaranteed, i.e., the existence of conjugate points along γ_1 are excluded. As opposed

to this, by making use of an analogue of the above indirect argument, valid for the timelike case, the existence of conjugate points along γ_1 could also be shown in the null case. This contradictory situation can only be avoided if the map ψ is one-to-one on $\psi[\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}] \subset M$ as we intended to show. \square

In virtue of Proposition 3.1 and Theorem 3.1 if γ does not terminate on a topological singularity the local diffeomorphism ψ is, in fact, guaranteed to be a diffeomorphism between $\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}$ and $\psi[\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}] \subset M$. To simplify the notation applied above, hereafter, the interior, $\text{int}(\psi[\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}])$, of the image $\psi[\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}]$ of $\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}$ will be denoted by \mathcal{U} , similarly, we shall denote the Cartesian product $\varsigma_{t_0} \times (t_0, t_2 + \varepsilon)$, which is an open subset of \mathbb{R}^n , by \mathcal{U}^* . Notice that by the above applied construction the interior $\text{int}(\mathcal{U}_{[\sigma_{t_0}, \varepsilon]})$ of $\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}$ is a proper subset of \mathcal{U}^* . Finally, we shall also signify the restriction of the inverse of ψ to $\mathcal{U} = \text{int}(\psi[\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}])$ by ϕ , i.e., $\phi = \psi^{-1}|_{\mathcal{U} = \text{int}(\psi[\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}])}$. This is, in fact, the map providing the imbedding $\phi : \mathcal{U} \rightarrow \mathcal{U}^*$ of the open subset $\mathcal{U} \subset M$ into the open subset \mathcal{U}^* of \mathbb{R}^n what we shall need in constructing our intermediate extension.

Notice that \mathcal{U} and \mathcal{U}^* are chosen so that the members of the congruence Γ in \mathcal{U} are represented by straight coordinate lines in $\phi[\mathcal{U}]$, and also that for any member $\bar{\gamma}$ of Γ which starts at a point of σ_{t_0} , and which is incomplete and future non-extendible in (M, g_{ab}) the curve $\phi \circ \bar{\gamma}$ can be continued straightly into the region $\mathcal{U}^* \setminus \phi[\mathcal{U}]$.

4 Extension of the spacetime metric

In order to show the existence of the desired ‘intermediate’ extension we also need to demonstrate that the spacetime metric can be extended from $\phi[\mathcal{U}] \subset \mathbb{R}^n$ to $\mathcal{U}^* \setminus \phi[\mathcal{U}] \subset \mathbb{R}^n$. Before doing this we recall some notions and results we shall apply.

Following the terminology introduced by Whitney [63, 64] a point set $\mathcal{A} \subset \mathbb{R}^n$ is said to possess property \mathcal{P} if there is a positive real number ω such that for any two points x and y of \mathcal{A} can be joined by a curve in \mathcal{A} of length $L \leq \omega \cdot \rho(x, y)$, where $\rho(x, y)$ denotes the Euclidean distance of the points $x, y \in \mathbb{R}^n$. The main results of Whitney concerning the extendibility of functions defined on a subset of \mathbb{R}^n , can be summarised as [63, 64]:

Theorem 4.1 *Assume that $\mathcal{A} \subset \mathbb{R}^n$ has property \mathcal{P} , and let $\mathcal{F}(x^1, \dots, x^n)$ be of class C^m , for some positive integer $m \in \mathbb{N}$, in \mathcal{A} . Suppose that $\ell \in \mathbb{N}$ is so that $\ell \leq m$, and also that each of the ℓ^{th} order derivatives $\partial_{x^1}^{\ell_1} \dots \partial_{x^n}^{\ell_n} \mathcal{F}$, with $\ell_1 + \dots + \ell_n = \ell$, can be defined on the boundary $\partial \mathcal{A}$ of \mathcal{A} so that they are continuous in $\overline{\mathcal{A}} = \mathcal{A} \cup \partial \mathcal{A}$. Then there exists an extension $\tilde{\mathcal{F}}$ of \mathcal{F} so that $\tilde{\mathcal{F}}$ is of class C^ℓ throughout \mathbb{R}^n . Moreover, the extension $\tilde{\mathcal{F}}$ can be chosen so that it is smooth (or even it can be guaranteed to be analytic) in $\mathbb{R}^n \setminus \overline{\mathcal{A}}$.*

Returning to the main line of our argument, next we shall prove that $\phi[\mathcal{U}]$ has property \mathcal{P} whenever the spacetime (M, g_{ab}) is globally hyperbolic. Recall first that according to the classical definition, see, e.g., Refs. [29, 40], a spacetime (M, g_{ab}) is said to be globally

hyperbolic if the following two conditions are satisfied. First, for arbitrary pairs of points $x, y \in M$ the intersection $J^+(x) \cap J^-(y)$ is compact. Second, (M, g_{ab}) is strongly causal, i.e., even the existence of almost closed causal curves is excluded. It was proved recently [5] that the second condition may be relaxed so that it suffices to exclude the existence of closed causal curves, i.e., to assume that (M, g_{ab}) is merely causal.

We would like to emphasise that the first condition, i.e., the compactness of the intersections $J^+(x) \cap J^-(y)$ for all $x, y \in M$, which, by excluding the existence of “naked singularities”, plays an important role in the argument below. In particular, it guarantees that whenever the points $x, y \in M$ are causally related so that $x \in J^-(y)$ then any future directed future inextendible timelike curve through x must intersect somewhere the boundary $\partial J^-(y)$ of the causal past of y . This is, in fact, the very property that ensures, whenever the spacetime is globally hyperbolic, the existence of those achronal curves which are constructed in the proof of the following proposition.

Proposition 4.1 *Suppose that the conditions of Theorem 3.1 hold. Assume, in addition, that the spacetime (M, g_{ab}) is globally hyperbolic. Then $\phi[\mathcal{U}] \subset \mathbb{R}^n$ has property \mathcal{P} .*

Proof: For the sake of some technical conveniences what will be shown below is, in fact, that $\phi[\mathcal{U}] \cup \varsigma_{t_0} \subset \mathbb{R}^n$ has property \mathcal{P} but then it follows straightforwardly that $\phi[\mathcal{U}] \subset \mathbb{R}^n$ does also possess property \mathcal{P} .

Let us start with the timelike case. In virtue of the freedom we have in selecting σ_{t_0} we may assume, without loss of generality, that σ_{t_0} is chosen so that $\varsigma_{t_0} = \phi[\sigma_{t_0}]$ is a convex subset of $\mathbb{R}^{n-1} \times \{t_0\}$. Let x and y be arbitrary points in \mathcal{U} . Denote by $\pi(x)$ and $\pi(y)$ their projections to σ_{t_0} by the congruence Γ . Denote, furthermore, by $l(\phi(\pi(x)), \phi(\pi(y)))$ the straight line connecting $\phi(\pi(x))$ and $\phi(\pi(y))$ in ς_{t_0} . Consider now the two-dimensional timelike surface, $\mathcal{L}_{[\pi(x), \pi(y)]} = \pi^{-1}[\psi[l(\phi(\pi(x)), \phi(\pi(y)))]] \subset \mathcal{U}$, generated by the members of Γ starting at the points of curve $\ell_{[\pi(x), \pi(y)]} = \psi[l(\phi(\pi(x)), \phi(\pi(y)))]$ in σ_{t_0} . Then, the Euclidean distance $\rho(\phi(x), \phi(y))$ of the points $\phi(x), \phi(y) \in \phi[\mathcal{U}]$, measured with respect to the Gaussian coordinates (x^1, \dots, x^{n-1}, t) , satisfies the relation

$$\rho(\phi(x), \phi(y)) = \sqrt{(t_x - t_y)^2 + [\tilde{\rho}(\phi(\pi(x)), \phi(\pi(y)))]^2}, \quad (4.1)$$

where $\tilde{\rho}(\phi(\pi(x)), \phi(\pi(y)))$ denotes the Euclidean distance of the points $\phi(\pi(x))$ and $\phi(\pi(y))$ in $\varsigma_{t_0} \subset \mathbb{R}^{n-1} \times \{t_0\}$.

Our aim is now to show that for an arbitrary choice of $x, y \in \mathcal{U}$ there exists a curve λ connecting $\phi(x)$ and $\phi(y)$ in $\phi[\mathcal{U}] \cup \varsigma_{t_0} \subset \mathbb{R}^n$ so that its length is less than $\omega \cdot \rho(\phi(x), \phi(y))$, for some fixed positive real number ω that is independent of x and y . As for the relative position of x and y we have the following two possibilities. Either they are causally related or not. In the first case, without loss of generality, we may assume that x belongs to the causal past of y , i.e., $x \in J^-(y)$. Let then the curve $\tilde{\lambda}$ defined to be the composition of the two curves $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ selected as follows. Denote by γ_x the member of the congruence Γ through the point x , and choose $\tilde{\lambda}_1$ to be the segment of γ_x from x to $z = \partial J^-(y) \cap \gamma_x$.

Let, furthermore, $\tilde{\lambda}_2$ be chosen to be the curve, connecting y and z , which is determined by the intersection of $\partial J^-(y)$ and that of the two dimensional timelike surface $\mathcal{L}_{[\pi(x), \pi(y)]}$. Since (M, g_{ab}) is globally hyperbolic the intersection $z = \partial J^-(y) \cap \gamma_x$ and the curve $\tilde{\lambda}_2$ exist, and since $\tilde{\lambda}_2$ lies on the achronal boundary of a past set it is achronal and at least a locally Lipschitz curve [29]. Notice that, in virtue of Proposition 3.1 and Theorem 3.1 the Gaussian coordinates are well-defined in \mathcal{U} so that the neighbourhood \mathcal{O}_q may be replaced in the argument of Lemma 3.1 by the subset \mathcal{U} of M . Therefore, by the application of Lemma 3.1 to the curve $\tilde{\lambda}_2 \subset \partial J^-(y)$ in \mathcal{U} , it can be seen that for the Euclidean length of $\lambda = \lambda_1 \cup \lambda_2$, where the curves λ_1, λ_2 are defined to be the image of the curves $\tilde{\lambda}_1, \tilde{\lambda}_2$ by the map ϕ , we have

$$\begin{aligned} L(\lambda) &= L(\lambda_1) + L(\lambda_2) \leq (t_z - t_x) + \sqrt{1 + K^2} \cdot \tilde{\rho}(\phi(\pi(x)), \phi(\pi(y))) \\ &\leq (t_y - t_x) + \sqrt{1 + K^2} \cdot \rho(\phi(x), \phi(y)) \leq \left(1 + \sqrt{1 + K^2}\right) \cdot \rho(\phi(x), \phi(y)), \end{aligned} \quad (4.2)$$

where the positive number K is chosen so that the uniform bound on the metric tensor components $g_{\alpha\beta}$ can be given, as above in the proof of Lemma 3.1, in the form $|g_{\alpha\beta}| < K^2/(n-1)^2$.

In the other case, i.e., whenever neither $x \in J^-(y)$ nor $y \in J^-(x)$, start with the intersections $\ell_x = J^-(x) \cap \ell_{[\pi(x), \pi(y)]}$ and $\ell_y = J^-(y) \cap \ell_{[\pi(x), \pi(y)]}$. Then, ℓ_x and ℓ_y are non-empty segments of $\ell_{[\pi(x), \pi(y)]}$ since $\pi(x) \in \ell_x$ and $\pi(y) \in \ell_y$. Consider now the two dimensional timelike surfaces $\mathcal{L}_x = \pi^{-1}[\ell_x]$ and $\mathcal{L}_y = \pi^{-1}[\ell_y]$, i.e., \mathcal{L}_x and \mathcal{L}_y are comprised by the members of Γ starting at the points of curves ℓ_x and ℓ_y in σ_{t_0} , respectively. Define, now, $\tilde{\lambda}_x$ and $\tilde{\lambda}_y$ to be the curves $\tilde{\lambda}_x = \mathcal{L}_x \cap \partial J^-(x)$ and $\tilde{\lambda}_y = \mathcal{L}_y \cap \partial J^-(y)$, respectively. Again, because (M, g_{ab}) is globally hyperbolic the curves $\tilde{\lambda}_x$ and $\tilde{\lambda}_y$ exist, and since $\partial J^-(x)$ and $\partial J^-(y)$ are achronal boundaries of past sets the curves $\tilde{\lambda}_x$ and $\tilde{\lambda}_y$ are achronal and at least locally Lipschitz. We also have that $\tilde{\lambda}_x$ and $\tilde{\lambda}_y$ either intersect at certain point $z \in \mathcal{U} \cup \sigma_{t_0}$ or not according to whether the segments ℓ_x and ℓ_y of the curve $\ell_{[\pi(x), \pi(y)]}$ overlap in σ_{t_0} or not.

Assume first that there exists a point $z \in \mathcal{U} \cup \sigma_{t_0}$ so that $\tilde{\lambda}_x$ and $\tilde{\lambda}_y$ intersect at z . Define, now, λ to be the combination $\lambda = \lambda_x \cup \lambda_y$, where λ_x is the segment of the curve $\phi[\tilde{\lambda}_x]$ from $\phi(x)$ to $\phi(z)$, and λ_y is the segment of the curve $\phi[\tilde{\lambda}_y]$ from $\phi(z)$ to $\phi(y)$. Then, for the Euclidean length of λ in \mathbb{R}^n

$$\begin{aligned} L(\lambda) &= L(\lambda_x) + L(\lambda_y) \leq \sqrt{1 + K^2} \cdot (\tilde{\rho}(\phi(\pi(x)), \phi(\pi(z))) + \tilde{\rho}(\phi(\pi(z)), \phi(\pi(y)))) \\ &\leq \sqrt{1 + K^2} \cdot \rho(\phi(x), \phi(y)) \end{aligned} \quad (4.3)$$

holds, where the real number $K > 0$ is defined as above.

Suppose, now, that $\tilde{\lambda}_x$ and $\tilde{\lambda}_y$ do not intersect in $\mathcal{U} \cup \sigma_{t_0}$, furthermore, denote by z_1 and z_2 the intersections $\tilde{\lambda}_x \cap \ell_{[\pi(x), \pi(y)]}$ and $\tilde{\lambda}_y \cap \ell_{[\pi(x), \pi(y)]}$, respectively. Denote, furthermore, by $\lambda_{z_1 z_2}$ the straight line segment of $l(\phi(\pi(x)), \phi(\pi(y)))$ connecting $\phi(z_1)$ and $\phi(z_2)$ in $\varsigma_{t_0} \subset \mathbb{R}^{n-1} \times \{t_0\}$. Then, by exactly the same type of reasoning that has already been applied above twice it can be justified that for the Euclidean length of $\lambda = \lambda_x \cup \lambda_{z_1 z_2} \cup \lambda_y$, where

$$\lambda_x = \phi[\tilde{\lambda}_x] \text{ and } \lambda_y = \phi[\tilde{\lambda}_y],$$

$$\begin{aligned} L(\lambda) &= L(\lambda_x) + L(\lambda_{z_1 z_2}) + L(\lambda_y) \\ &\leq \sqrt{1 + K^2} \cdot \tilde{\rho}(\phi(\pi(x)), \phi(z_1)) + \tilde{\rho}(\phi(z_1), \phi(z_2)) + \sqrt{1 + K^2} \cdot \tilde{\rho}(\phi(z_2), \phi(\pi(y))) \\ &\leq \sqrt{1 + K^2} \cdot \rho(\phi(x), \phi(y)) \end{aligned} \tag{4.4}$$

holds, where the real number $K > 0$ is defined as above.

Finally, in virtue of (4.2), (4.3) and (4.4) we have then that in either case the Euclidean length $L(\lambda)$ is definitely smaller than or equal to $\omega = 1 + \sqrt{1 + K^2}$ times the Euclidean distance $\rho(\phi(x), \phi(y))$ between $\phi(x)$ and $\phi(y)$.

The proof for the null case can be derived analogously. The most significant technical differences that arise in the argument are rooted in the following facts. In order to prove Lemma 3.2—which is the counterpart of Lemma 3.1 applied above—the auxiliary coordinates $(\bar{x}^1, \dots, \bar{x}^{n-2}, \bar{r}, \bar{t})$, along with the \bar{t} -coordinate lines, which comprise an $n - 1$ -parameter congruence of timelike curves $\bar{\Gamma}$, as well as, the space of these timelike curves, i.e., the screening hypersurface $\bar{\sigma}_q$, were introduced merely locally in \mathcal{O}_q . Notice, however, that since, in virtue of Theorem 3.1, the Gaussian null coordinates are guaranteed to be well-defined on the entire of \mathcal{U} the auxiliary coordinates $(\bar{x}^1, \dots, \bar{x}^{n-2}, \bar{r}, \bar{t})$, given by the relations (3.16), become also well-defined everywhere on \mathcal{U} . Similarly, the screening hypersurface σ_{t_0} , applied above in the timelike case, may be replaced by the space of the \bar{t} -coordinate lines, denoted by $\bar{\sigma}_{\mathcal{U}}$, in the present case. Notice that, whenever (M, g_{ab}) is globally hyperbolic, $\bar{\sigma}_{\mathcal{U}}$ as a point set, can be represented by the intersection $J^-[\mathcal{U}] \cap \partial\mathcal{U}$. Finally, by noticing that the relations (3.16) imply that the Euclidean length of any curve in $\phi[\mathcal{U}]$ is the same regardless it is measured in terms of the Gaussian null coordinates $(x^1, \dots, x^{n-2}, r, t)$ or in terms of the auxiliary coordinates $(\bar{x}^1, \dots, \bar{x}^{n-2}, \bar{r}, \bar{t})$ —this invariance property, in particular, also implies that for any $x, y \in \mathcal{U}$ the distance $\bar{\rho}(\bar{\gamma}_x, \bar{\gamma}_y)$ of $\bar{\gamma}_x$ and $\bar{\gamma}_y$ in $\bar{\sigma}_{\mathcal{U}}$ has to be less than equal to the Euclidean distance of $\phi(x)$ and $\phi(y)$ as determined with respect to the Gaussian null coordinates—it can be justified that the assertion of our proposition does also hold in the null case. \square

We would like to mention that the condition used in the above proposition is sufficient but it is not necessary. More specifically, it is not hard to construct a spacetime with a subset $\phi[\mathcal{U}]$ of \mathbb{R}^n which does possess the property \mathcal{P} in spite of the fact the associated spacetime is not globally hyperbolic. As an immediate example of this type one may think of a spacetime yielded by the removal of half of a causal geodesic from the Minkowski spacetime with dimension $n \geq 3$ which is not globally hyperbolic, in fact, it is not even causally simple.

To prove the existence of a C^{k-} extension of the smooth spacetime metric from $\phi[\mathcal{U}]$ we shall need the following proposition, the assumptions of which are reminiscent of that of Proposition 3.3.1 of [42].

Proposition 4.2 *Let $\phi[\mathcal{U}] \subset \mathbb{R}^n$ be as defined above and assume that it has the property \mathcal{P} . Suppose that $\mathcal{F} : \phi[\mathcal{U}] \rightarrow \mathbb{R}$ is a C^k function, and also that its k^{th} order derivatives $\partial_{x_1}^{k_1} \cdots \partial_{x_n}^{k_n} \mathcal{F}$, with $k_1 + \cdots + k_n = k \geq 0$, are bounded on $\phi[\mathcal{U}]$. Then the $(k-1)^{\text{th}}$ order derivatives of \mathcal{F} are Lipschitz functions on the closure $\overline{\phi[\mathcal{U}]}$ of $\phi[\mathcal{U}]$. Moreover, \mathcal{F} extends to $\overline{\phi[\mathcal{U}]}$ so that its extension $\tilde{\mathcal{F}}$ is a function of class C^{k-} throughout $\overline{\phi[\mathcal{U}]}$.*

Proof: To justify that the $(k-1)^{\text{th}}$ order derivatives of \mathcal{F} are Lipschitz functions on $\phi[\mathcal{U}]$ it suffices to recall the first part of the proof of Proposition 3.3.1 of [42].

To see that the $(k-1)^{\text{th}}$ order derivatives of \mathcal{F} are also Lipschitz functions on the closure $\overline{\phi[\mathcal{U}]}$ of $\phi[\mathcal{U}]$ notice that to any pair of points $r, s \in \overline{\phi[\mathcal{U}]} \subset \mathbb{R}^n$ there must exist point sequences $\{r_i\}$ and $\{s_i\}$, consisting of points of $\phi[\mathcal{U}]$, which converge to r and s , respectively. Hence, for any value of the index i the inequality

$$\left| \partial_{x_1}^{k_1} \cdots \partial_{x_n}^{k_n} \mathcal{F}(r_i) - \partial_{x_1}^{k_1} \cdots \partial_{x_n}^{k_n} \mathcal{F}(s_i) \right| < \mathcal{K} \cdot \rho(r_i, s_i) \quad (4.5)$$

holds, where \mathcal{K} is the Lipschitz constant bounding the absolute value of the difference of the $(k-1)^{\text{th}}$ order derivatives of \mathcal{F} on $\phi[\mathcal{U}]$. Notice that then the $(k-1)^{\text{th}}$ order derivatives of \mathcal{F} are also guaranteed to be uniformly continuous on $\phi[\mathcal{U}]$, so the extensions of the $(k-1)^{\text{th}}$ order derivatives of \mathcal{F} to the boundary $\overline{\phi[\mathcal{U}]} \setminus \phi[\mathcal{U}]$ are unique. Thereby, for any real number $\mathcal{K}' > \mathcal{K}$ the inequality

$$\left| \partial_{x_1}^{k_1} \cdots \partial_{x_n}^{k_n} \mathcal{F}(r) - \partial_{x_1}^{k_1} \cdots \partial_{x_n}^{k_n} \mathcal{F}(s) \right| < \mathcal{K}' \cdot \rho(r, s), \quad (4.6)$$

must hold in the limiting case, as well, which implies then that the $(k-1)^{\text{th}}$ order derivatives of \mathcal{F} are Lipschitz functions on $\overline{\phi[\mathcal{U}]}$.

Finally, to justify that there exists an extension $\tilde{\mathcal{F}}$ of \mathcal{F} which is a function of class C^{k-} on $\overline{\phi[\mathcal{U}]}$ we may refer to Lemma 4 of [63] and to the fact that $\phi[\mathcal{U}]$ is assumed to have property \mathcal{P} . It follows then that for arbitrary choices of the sub-orders k_1, \dots, k_n with $0 \leq k_1 + \cdots + k_n = l \leq k-1$ the l^{th} order derivatives $\partial_{x_1}^{k_1} \cdots \partial_{x_n}^{k_n} \mathcal{F}$ are uniformly continuous which completes our proof. \square

We would like to emphasise that property \mathcal{P} plays a crucial role in the above argument. First of all, whenever property \mathcal{P} is not guaranteed to be satisfied, as it is demonstrated by the example given in footnote 3 of Ref. [63], it is possible to construct a function so that all of its partial derivatives are continuous on $\overline{\phi[\mathcal{U}]}$ whereas at certain points of the boundary the function itself is not continuous. A more elementary way of demonstrating that the assertion of Proposition 4.2 is manifestly false without the use of property \mathcal{P} can be given as follows.

Example 4.1 *Choose $\phi[\mathcal{U}]$ to be an open subset of \mathbb{R}^2 , with coordinates (x, t) , given as*

$$\phi[\mathcal{U}] = (-1, 1) \times (-1, 1) \setminus \{0\} \times [0, 1). \quad (4.7)$$

Clearly, $\phi[\mathcal{U}]$ does not possess property \mathcal{P} . Consider now the function \mathcal{F} on $\phi[\mathcal{U}]$ defined as

$$\mathcal{F} = \begin{cases} \exp(-\frac{1}{t}), & \text{if } x > 0 \text{ and } t \in (0, 1); \\ -\exp(-\frac{1}{t}), & \text{if } x < 0 \text{ and } t \in (0, 1); \\ 0, & \text{otherwise.} \end{cases} \quad (4.8)$$

Then, \mathcal{F} is a C^∞ function on $\phi[\mathcal{U}]$, and the partial derivatives of \mathcal{F} , up to any fixed order, are uniformly bounded there. Nevertheless, it is straightforward to justify that \mathcal{F} cannot even have a continuous extension to the closure $\overline{\phi[\mathcal{U}]} = [-1, 1] \times [-1, 1]$ of $\phi[\mathcal{U}]$.

As we have just seen for a C^k function $\mathcal{F} : \phi[\mathcal{U}] \rightarrow \mathbb{R}$ satisfying the conditions of the above proposition each of the derivatives $\partial_{x^1}^{k_1} \cdots \partial_{x^n}^{k_n} \mathcal{F}$, with $k_1 + \cdots + k_n = k - 1 \geq 0$, are uniformly continuous functions in $\overline{\phi[\mathcal{U}]}$. Then, in virtue of Theorem 4.1 and Proposition 4.2, \mathcal{F} can be extended to $\mathcal{U}^* \setminus \phi[\mathcal{U}] \subset \mathbb{R}^n$ so that its extension $\tilde{\mathcal{F}}$ is of class C^{k-} throughout \mathcal{U}^* . Thus, concerning our specific problem it follows that a smooth Lorentzian metric g_{ab} is guaranteed to have a C^{k-} extension from $\phi[\mathcal{U}]$ to $\mathcal{U}^* \subset \mathbb{R}^n$ if all the derivatives $\partial_{x^1}^{k_1} \cdots \partial_{x^n}^{k_n} g_{\alpha\beta}$, with $k_1 + \cdots + k_n = k \geq 0$, are guaranteed to be uniformly bounded along the members of Γ .

Before examining the boundedness of the derivatives $\partial_{x^1}^{k_1} \cdots \partial_{x^n}^{k_n} g_{\alpha\beta}$, with $k_1 + \cdots + k_n = k \geq 0$, let us introduce the following notations. Denote by $\{E_{(\alpha)}^a\}$ the coordinate basis field $E_{(\alpha)}^a := (\partial/\partial x^\alpha)^a$ associated with a local coordinate system (x^1, \dots, x^n) in \mathcal{U} . Moreover, denote by $\nabla_{(\varepsilon)}$ the covariant derivative with respect to the vector field $E_{(\varepsilon)}^e$, i.e., $\nabla_{(\varepsilon)} := E_{(\varepsilon)}^e \nabla_e$, where ∇_a stands for the unique torsion free metric compatible covariant derivative operator. Then the relevant coordinate components of the metric and their partial derivatives read as

$$\begin{aligned} g_{\alpha\beta} &= g_{ab} E_{(\alpha)}^a E_{(\beta)}^b \\ \partial_{x^{\varepsilon_1}} g_{\alpha\beta} &= g_{ab} \left[\left(\nabla_{(\varepsilon_1)} E_{(\alpha)}^a \right) E_{(\beta)}^b + E_{(\alpha)}^a \left(\nabla_{(\varepsilon_1)} E_{(\beta)}^b \right) \right] \\ &\vdots \\ \partial_{x^{\varepsilon_l}} \cdots \partial_{x^{\varepsilon_1}} g_{\alpha\beta} &= g_{ab} \left[\left(\nabla_{(\varepsilon_l)} \cdots \nabla_{(\varepsilon_1)} E_{(\alpha)}^a \right) E_{(\beta)}^b + \left(\nabla_{(\varepsilon_{l-1})} \cdots \nabla_{(\varepsilon_1)} E_{(\alpha)}^a \right) \left(\nabla_{(\varepsilon_l)} E_{(\beta)}^b \right) \right. \\ &\quad \left. + \cdots + \left(\nabla_{(\varepsilon_l)} E_{(\alpha)}^a \right) \left(\nabla_{(\varepsilon_{l-1})} \cdots \nabla_{(\varepsilon_1)} E_{(\beta)}^b \right) + E_{(\alpha)}^a \left(\nabla_{(\varepsilon_l)} \cdots \nabla_{(\varepsilon_1)} E_{(\beta)}^b \right) \right]. \end{aligned} \quad (4.9)$$

Notice that the right hand sides of the higher order derivatives as they appear in (4.9) are not manifestly symmetric in the indices $\varepsilon_l, \dots, \varepsilon_1$ as they should be according to the expressions on the left hand sides. Nevertheless, by making use of the fact that the local coordinate basis fields $E_{(\alpha)}^a$ commute, $[E_{(\alpha)}^a, E_{(\beta)}^b] = 0$, along with the symmetry properties of the curvature tensor, the desired symmetry relations can be shown to be satisfied by the right hand sides.

Since \mathcal{U} is covered by the members of Γ , in virtue of the above argument, it suffices to show that terms of the form $g_{ab}(\nabla_{(\varepsilon_l)} \cdots \nabla_{(\varepsilon_1)} E_{(\alpha)}^a)(\nabla_{(\rho_m)} \cdots \nabla_{(\rho_1)} E_{(\beta)}^b)$, with $0 \leq l, m$ and $l + m \leq k$, are uniformly bounded along the members of the synchronised congruence

Γ . These terms are guaranteed to be uniformly bounded if the components of the vector fields $\nabla_{(\varepsilon_l)} \cdots \nabla_{(\varepsilon_1)} E_{(\alpha)}^a$ for all values $l \leq k$ remain uniformly bounded with respect to a synchronised orthonormal (resp., pseudo-orthonormal) frame field $\{e_{(a)}^a\}$ along the members of Γ . The behaviour of the vector fields $\nabla_{(\varepsilon_l)} \cdots \nabla_{(\varepsilon_1)} E_{(\alpha)}^a$ had already been examined in [42] and it was found there that along the members of causal geodesic congruences it is determined by a generalised form of the Jacobi equation. More precisely, as it was justified by Proposition 3.3.3 of [42], a vector field of the form $\nabla_{(\varepsilon_l)} \cdots \nabla_{(\varepsilon_1)} E_{(\alpha)}^a$ does satisfy, along the causal geodesics belonging to the congruence Γ , the *generalised Jacobi equation*

$$\nabla_{(n)} \nabla_{(n)} \left(\nabla_{(\varepsilon_l)} \cdots \nabla_{(\varepsilon_1)} E_{(\alpha)}^a \right) = R_{efh}{}^a E_{(n)}^e \left(\nabla_{(\varepsilon_l)} \cdots \nabla_{(\varepsilon_1)} E_{(\alpha)}^f \right) E_{(n)}^h + \left[\mathcal{Q}_{(l)} \right]_{(\alpha)}^a, \quad (4.10)$$

where the last term on the r.h.s., i.e., the relevant form of the vector field $\left[\mathcal{Q}_{(l)} \right]_{(\alpha)}^a$, is given recursively by the following relations

$$\begin{aligned} \left[\mathcal{Q}_{(l)} \right]_{(\alpha)}^a = & \\ \nabla_{(n)} \left[R_{efh}{}^a E_{(n)}^e E_{(\varepsilon_l)}^f \left(\nabla_{(\varepsilon_{l-1})} \cdots \nabla_{(\varepsilon_1)} E_{(\alpha)}^h \right) \right] &+ R_{efh}{}^a E_{(n)}^e E_{(\varepsilon_l)}^f \left[\nabla_{(n)} \left(\nabla_{(\varepsilon_{l-1})} \cdots \nabla_{(\varepsilon_1)} E_{(\alpha)}^h \right) \right] \\ &+ \nabla_{(\varepsilon_l)} \left[R_{efh}{}^a E_{(n)}^e \left(\nabla_{(\varepsilon_{l-1})} \cdots \nabla_{(\varepsilon_1)} E_{(\alpha)}^f \right) E_{(n)}^h \right] + \nabla_{(\varepsilon_l)} \left[\mathcal{Q}_{(l-1)} \right]_{(\alpha)}^a, \end{aligned} \quad (4.11)$$

while

$$\left[\mathcal{Q}_{(0)} \right]_{(\alpha)}^a = 0. \quad (4.12)$$

By making use of the generalised Jacobi equation, along with Proposition 3.3.4, Corollary 3.3.5, Lemma 3.3.6 and Proposition 3.3.7 of [42], it can be shown that the components of the vector fields of the form $\nabla_{(\varepsilon_l)} \cdots \nabla_{(\varepsilon_1)} E_{(\alpha)}^a$, with $0 \leq l \leq k$, with respect to a synchronised orthonormal (resp., pseudo-orthonormal) basis field $\{e_{(a)}^a\}$ on \mathcal{U} , are bounded whenever the components of the Riemann tensor

$$R_{\mathbf{a}\mathbf{b}\mathbf{c}\mathbf{d}} = R_{abcd} e_{(\mathbf{a})}^a e_{(\mathbf{b})}^b e_{(\mathbf{c})}^c e_{(\mathbf{d})}^d, \quad (4.13)$$

along with the components of the covariant derivatives of the Riemann tensor

$$\nabla_{\mathbf{h}_l} \cdots \nabla_{\mathbf{h}_1} R_{\mathbf{a}\mathbf{b}\mathbf{c}\mathbf{d}} = e_{(\mathbf{h}_l)}^{h_l} \cdots e_{(\mathbf{h}_1)}^{h_1} (\nabla_{h_l} \cdots \nabla_{h_1} R_{abcd}) e_{(\mathbf{a})}^a e_{(\mathbf{b})}^b e_{(\mathbf{c})}^c e_{(\mathbf{d})}^d \quad (4.14)$$

up to order $0 \leq l \leq k$ are bounded with respect to a synchronised orthonormal (resp., pseudo-orthonormal) basis field $\{e_{(\mathbf{a})}^a\}$ on \mathcal{U} .

These requirements, however, turned out to be too restrictive, which—in virtue of the argument below—means that the conditions of Proposition 3.3.4 and Corollary 3.3.5 of [42] can be relaxed. In particular, the assertion of Proposition 3.3.4 of [42] remains intact if instead of requiring the norm $q^{(k)}(t) = \left\| \left[\mathcal{Q}_{(k)} \right]_{(\alpha)}^a \right\| |\bar{\gamma}(t)|$ of the source term in the generalised Jacobi equation to be uniformly bounded along a member $\bar{\gamma}$ of Γ we merely demand that it does not blow up too fast in the sense that its line integral remains finite along $\bar{\gamma}$. Recall

that the norm $\|X^a\|$ of a vector field X^a , with respect to a synchronised basis field $\{e_{(a)}^a\}$ and a Lorentzian metric g_{ab} , was defined as

$$\|X^a\| = \sqrt{\sum_{b=1}^n [g_{ab} X^a e_{(b)}^b]^2}. \quad (4.15)$$

The key point in the argument ensuring that this replacement can be done is that the proof of Proposition 3.3.4 of [42], which is yielded by a generalisation of that of Proposition 3.1 of [14], remains valid provided that the line integral of the source term is guaranteed to be finite. In particular, as an immediate generalisation of Corollary 3.3.5 of [42], it can be seen that whenever there exists a positive real number $r_0 > 0$ so that $\|R_{abcd}\| = \sqrt{\sum_{a,b,c,d=1}^n [R_{abcd}]^2} \leq r_0$ along $\bar{\gamma}$ then the inequality also holds

$$\begin{aligned} \|\nabla_{(\varepsilon_k)} \cdots \nabla_{(\varepsilon_1)} E_{(\alpha)}^a|_{\bar{\gamma}(t)}\| &\leq \|\nabla_{(\varepsilon_k)} \cdots \nabla_{(\varepsilon_1)} E_{(\alpha)}^a|_{\bar{\gamma}(t_0)}\| \cdot \cosh[\sqrt{r_0}(t - t_0)] \\ &\quad + \frac{\|\nabla_{(n)} \nabla_{(\varepsilon_k)} \cdots \nabla_{(\varepsilon_1)} E_{(\alpha)}^a|_{\bar{\gamma}(t_0)}\|}{\sqrt{r_0}} \cdot \sinh[\sqrt{r_0}(t - t_0)] \\ &\quad + \frac{1}{2} \left[\frac{e^{\sqrt{r_0}t}}{\sqrt{r_0}} \int_{t_0}^t e^{-\sqrt{r_0}t'} q^{(k)}(t') dt' - \frac{e^{-\sqrt{r_0}t}}{\sqrt{r_0}} \int_{t_0}^t e^{\sqrt{r_0}t'} q^{(k)}(t') dt' \right] \end{aligned} \quad (4.16)$$

holds along $\bar{\gamma}$. Noticing then that the functions $e^{\pm\sqrt{r_0}t}$ remain bounded on the finite interval $[t_0, t_2]$ it is straightforward to see that the last term remains bounded along $\bar{\gamma}$ —and, in turn, the components of the vector field $\nabla_{(\varepsilon_k)} \cdots \nabla_{(\varepsilon_1)} E_{(\alpha)}^a$ are also bounded there—provided that the integral $\int_{t_0}^{t_2} q^{(k)}(t) dt$ is guaranteed to be finite. In virtue of (4.11) the source term $q^{(k)}$ is given in terms of expressions containing the components of the k^{th} -order covariant derivatives of the Riemann tensor and of lower order terms. Combining this with a suitable adaptation of the proof of Proposition 3.3.7 of [42] it can be shown that the integral $\int_{t_0}^{t_2} q^{(k)}(t) dt$ is guaranteed to be finite along $\bar{\gamma}$ whenever the components of the Riemann tensor R_{abcd} , along with the components of the covariant derivatives of the Riemann tensor, $\nabla_{h_l} \cdots \nabla_{h_1} R_{abcd}$, up to order $0 \leq l \leq k-1$ are bounded, and also the line integrals of the components of the k^{th} -order covariant derivatives of the Riemann tensor, $\nabla_{h_k} \cdots \nabla_{h_1} R_{abcd}$, are finite along $\bar{\gamma}$.

By applying the above outlined argument to the individual members of the $(n-1)$ -parameter synchronised family of causal geodesics Γ simultaneously—and by making also use of the fact that σ_{t_0} was chosen to be a subset of Σ so that the closure of σ_{t_0} is compact in Σ —it can be justified that the components of the vector fields of the form $\nabla_{(\varepsilon_l)} \cdots \nabla_{(\varepsilon_1)} E_{(\alpha)}^a$, with $0 \leq l \leq k$, are uniformly bounded along the members of Γ whenever the components of the Riemann tensor R_{abcd} , along with the components of the covariant derivatives of the Riemann tensor, $\nabla_{h_l} \cdots \nabla_{h_1} R_{abcd}$, up to order $0 \leq l \leq k-1$ are uniformly bounded, and also the line integrals of the components of the k^{th} -order covariant derivatives of the Riemann tensor remain finite along the members of Γ , where all the components of the Riemann tensor and its covariant derivatives are meant to be measured with respect to a synchronised basis field defined along the members of Γ .

Finally, by combining all the above results it is straightforward to see that whenever the components of the vector fields $\nabla_{(\varepsilon_l)} \cdots \nabla_{(\varepsilon_1)} E_{(\alpha)}^a$, with $0 \leq l \leq k$, and with respect to

a synchronised orthonormal (resp., pseudo-orthonormal) basis field $\{e_{(a)}^a\}$, are guaranteed to be uniformly bounded along the members of the congruence Γ on \mathcal{U} then it is also guaranteed that their inner products $g_{ab}(\nabla_{(\varepsilon_l)} \cdots \nabla_{(\varepsilon_1)} E_{(\alpha)}^a)(\nabla_{(\rho_m)} \cdots \nabla_{(\rho_1)} E_{(\beta)}^b)$, with $0 \leq l, m$ and $l + m \leq k$, are bounded along the members of the congruence Γ on \mathcal{U} , and also that according to (4.9), the partial derivatives $\partial_{x_1}^{k_1} \cdots \partial_{x_n}^{k_n} g_{\alpha\beta}$ of the metric, with sub-orders $0 \leq k_1 + \cdots + k_n = k$ have to be bounded in $\phi[\mathcal{U}]$. Consequently, these k^{th} -order derivatives are bounded on $\phi[\mathcal{U}]$ whenever the components R_{abcd} of the Riemann tensor, along with the components $\nabla_{h_l} \cdots \nabla_{h_1} R_{abcd}$ of its covariant derivatives up to order $0 \leq l \leq k - 1$ are guaranteed to be uniformly bounded, and moreover the line integrals of the components of the k^{th} -order covariant derivatives, $\nabla_{h_k} \cdots \nabla_{h_1} R_{abcd}$, are finite along the members of Γ on \mathcal{U} , where the components of the Riemann tensor and its covariant derivatives are meant to be registered with respect to a synchronised basis field defined along the members of Γ .

The above argument, along with Theorem 4.1, Proposition 4.1 and Proposition 4.2, provides then the justification of the following.

Theorem 4.2 *Let $\gamma : (t_1, t_2) \rightarrow M$ be an incomplete non-extendible timelike (resp., null) geodesic curve which does not terminate either on a tidal force tensor singularity or on a topological singularity. Let Γ, \mathcal{U} and \mathcal{U}^* be chosen as they were in section 3. Suppose, finally, that (M, g_{ab}) is globally hyperbolic and the components R_{abcd} of the Riemann tensor, along with the components $\nabla_{\epsilon_l} \cdots \nabla_{\epsilon_1} R_{abcd}$ of its covariant derivatives up to order $0 \leq l \leq k - 1$ are bounded on \mathcal{U} , and also the line integrals of the components of the k^{th} -order covariant derivatives, $\nabla_{h_k} \cdots \nabla_{h_1} R_{abcd}$, are finite along the members of Γ , where all the components are meant to be measured with respect to a synchronised orthonormal (resp., pseudo-orthonormal) basis field $\{e_{(a)}^a\}$ on \mathcal{U} . Then, there exists a C^{k-} extension $\phi : (\mathcal{U}, g_{ab}|_{\mathcal{U}}) \rightarrow (\mathcal{U}^*, g_{ab}^*)$ of the subspacetime $(\mathcal{U}, g_{ab}|_{\mathcal{U}})$ into a spacetime $(\mathcal{U}^*, g_{ab}^*)$ so that for any member $\bar{\gamma}$ of Γ starting on σ_{t_0} and which is incomplete and non-extendible in (M, g_{ab}) the timelike (resp., null) geodesic curve $\phi \circ \bar{\gamma}$ is extendible in $(\mathcal{U}^*, g_{ab}^*)$.*

We would like to emphasise that whenever an extension $(\mathcal{U}^*, g_{ab}^*)$ of the spacetime $(\mathcal{U}, g_{ab}|_{\mathcal{U}})$ exists the limit of $\phi^* g_{ab}$ on the boundary $\partial(\phi[\mathcal{U}])$ of $\phi[\mathcal{U}]$ in \mathcal{U}^* must be uniquely determined. As opposed to this—and as a direct consequence of the fact that we have not imposed any sort of restriction on g_{ab}^* , e.g., in terms of certain field equations, that could reduce generality—the metric g_{ab}^* is by no means unique on $\mathcal{U}^* \setminus \overline{\phi[\mathcal{U}]}$. Nevertheless, by making use, for instance, of the results of Whitney's, see Lemma 2 and Theorem I of [64], the metric g_{ab}^* can be guaranteed to be smooth or even to be analytic everywhere on $\mathcal{U}^* \setminus \overline{\phi[\mathcal{U}]}$.

5 Topological singularities

This section is to characterise spacetimes with a topological singularity. To start off, assume first that the conditions of Proposition 3.1 hold. Accordingly, let $\gamma : (t_1, t_2) \rightarrow M$ be an incomplete non-extendible causal geodesic in M , which does not terminate on a tidal force tensor singularity. We shall assume that a particular choice for $t_0 \in (t_1, t_2)$, $\sigma_{t_0} \subset \Sigma$, Γ, ε

and, thereby, for $\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}$ has been made, and, in virtue of the assertion of Proposition 3.1, the map $\psi : \mathcal{U}_{[\sigma_{t_0}, \varepsilon]} \rightarrow M$ is guaranteed to be a local diffeomorphism. Finally, we shall suppose that $\gamma : (t_1, t_2) \rightarrow M$ *does* terminate on a topological singularity. Recall that the last assumption implies that for any choice of $t_0 \in (t_1, t_2)$, σ_{t_0} and ε the subset $\text{int}(\psi[\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}]) \subset M$ —which in this section will also be denoted by \mathcal{U} —is *not* simply connected, therefore it is not simply connected for our particular choice either.

Consider now the universal cover \widetilde{M} of M and denote by $\mathcal{C} : \widetilde{M} \rightarrow M$ the associated covering map. Since, by construction, $\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}$ is a connected subset of \mathbb{R}^n , and $\psi : \mathcal{U}_{[\sigma_{t_0}, \varepsilon]} \rightarrow M$ is a smooth map, there always exists exactly one lift $\widetilde{\psi}$ of ψ through \mathcal{C} such that if for an arbitrarily chosen $q \in M$ we have that whenever $\psi(x^\alpha(q)) = \mathcal{C}(\widetilde{q})$ is satisfied the relation $\widetilde{\psi}(x^\alpha(q)) = \widetilde{q}$ also holds (see, e.g., Appendix A of [37]). This, in particular, guarantees that once one of the ‘pre-images’ $\widetilde{p} \in \widetilde{M}$ of $p = \gamma(t_0)$ was chosen there is a unique way to determine $\widetilde{\sigma}_{t_0}$, $\widetilde{\Gamma}$ and $\widetilde{\mathcal{U}}_{[\widetilde{\sigma}_{t_0}, \widetilde{\varepsilon}]}$ so that they are related to each other by exactly the same construction as σ_{t_0} , Γ and $\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}$ were in the previous sections.

Consider now the spacetime $(\widetilde{M}, \widetilde{g}_{ab})$ where the metric \widetilde{g}_{ab} is defined to be the ‘pull-back’ of g_{ab} by the derivative, \mathcal{C}^* , of the smooth map $\mathcal{C} : \widetilde{M} \rightarrow M$. Notice that the above construction ensures that for the spacetime $(\widetilde{M}, \widetilde{g}_{ab})$ all the conditions of Proposition 3.1 are satisfied where now the causal geodesic $\widetilde{\gamma}$, the congruence $\widetilde{\Gamma}$ and $\widetilde{\mathcal{U}}_{[\widetilde{\sigma}_{t_0}, \widetilde{\varepsilon}]}$ are replacing the corresponding objects γ , Γ and $\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}$ applied in the previous sections. In addition, the above construction also guaranties that the causal geodesic $\widetilde{\gamma}$ does not terminate on a topological singularity in $(\widetilde{M}, \widetilde{g}_{ab})$. By utilising then the implications of Theorem 3.1 we have that the map $\widetilde{\psi} : \text{int}(\widetilde{\mathcal{U}}_{[\widetilde{\sigma}_{t_0}, \widetilde{\varepsilon}]}) \rightarrow \widetilde{M}$ does, in fact, act as a diffeomorphism between $\text{int}(\widetilde{\mathcal{U}}_{[\widetilde{\sigma}_{t_0}, \widetilde{\varepsilon}]})$ and $\widetilde{\mathcal{U}} = \widetilde{\psi}[\text{int}(\widetilde{\mathcal{U}}_{[\widetilde{\sigma}_{t_0}, \widetilde{\varepsilon}]})] \subset \widetilde{M}$. Finally, in accordance with the relevant results and notation introduced at the end of section 3, we shall denote by $\widetilde{\phi}$ the restriction of the inverse of $\widetilde{\psi}$ to $\widetilde{\mathcal{U}}$. Notice that since $\widetilde{\psi}$, by construction, is smooth $\widetilde{\phi} : \widetilde{\mathcal{U}} \rightarrow \text{int}(\widetilde{\mathcal{U}}_{[\widetilde{\sigma}_{t_0}, \widetilde{\varepsilon}]})$ is also a smooth map.¹²

According to the above described construction we may think of \mathcal{U} as the factor space $\widetilde{\mathcal{U}}/\mathcal{C}$, i.e., the points of \mathcal{U} can be represented by equivalence classes of the points of $\widetilde{\mathcal{U}}$ where $\widetilde{r}, \widetilde{s} \in \widetilde{\mathcal{U}}$ belong to the same class if $\mathcal{C}(\widetilde{r}) = \mathcal{C}(\widetilde{s})$. Moreover, since the spacetime (M, g_{ab}) cannot be causal whenever $(\widetilde{M}, \widetilde{g}_{ab})$ is not causal, and the “diamonds” $J^+(x) \cap J^-(y)$ for all $x, y \in M$ could not be compact if there were $\widetilde{x}, \widetilde{y} \in \widetilde{M}$ so that the diamond $\widetilde{J}^+(\widetilde{x}) \cap \widetilde{J}^-(\widetilde{y})$ would not be compact in \widetilde{M} we also have that the spacetime $(\widetilde{M}, \widetilde{g}_{ab})$ has to be globally hyperbolic whenever (M, g_{ab}) is globally hyperbolic. Similarly, whenever the components R_{abcd} of the Riemann tensor, along with the components $\nabla_{e_1} \dots \nabla_{e_l} R_{abcd}$ of its covariant derivatives up to order $0 \leq l \leq k-1$, are guaranteed to be uniformly bounded along the members of Γ , and moreover the line integral of the components of the k^{th} -order covariant derivatives, $\nabla_{h_k} \dots \nabla_{h_1} R_{abcd}$, are finite along the members of Γ , where all the components are meant to be measured with respect to a synchronised orthonormal

¹²Notice that, without loss of generality, the subsets $\widetilde{\mathcal{U}}_{[\widetilde{\sigma}_{t_0}, \widetilde{\varepsilon}]}$ and $\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}$ of \mathbb{R}^n could be identified. Nevertheless, we shall keep the above notation until the end of this argument.

(resp., pseudo-orthonormal) basis field $\{e_{(a)}^a\}$ on \mathcal{U} , according to the way the metric \tilde{g}_{ab} was constructed, we also have that the components $\tilde{R}_{abc\bar{d}}$ of the Riemann tensor, along with the components $\tilde{\nabla}_{\epsilon_l} \dots \tilde{\nabla}_{\epsilon_1} \tilde{R}_{abc\bar{d}}$ of its covariant derivatives up to order $0 \leq l \leq k-1$, are uniformly bounded along the members of $\tilde{\Gamma}$, and moreover the line integral of the components of the k^{th} -order covariant derivatives, $\tilde{\nabla}_{\epsilon_k} \dots \tilde{\nabla}_{\epsilon_1} \tilde{R}_{abc\bar{d}}$, are finite along the members of $\tilde{\Gamma}$, where all the components are meant to be measured with respect to a synchronised orthonormal (resp., pseudo-orthonormal) basis field $\{\tilde{e}_{(a)}^a\}$ on $\tilde{\mathcal{U}}$.

Thereby, if in addition to the assumptions that have been made above it is also assumed that (M, g_{ab}) is globally hyperbolic, and also that the components $R_{abc\bar{d}}$ of the Riemann tensor, along with the components $\nabla_{\epsilon_l} \dots \nabla_{\epsilon_1} R_{abc\bar{d}}$ of its covariant derivatives up to order $0 \leq l \leq k-1$, are guaranteed to be uniformly bounded along the members of Γ , and also the line integral of the components of the k^{th} -order covariant derivatives, $\nabla_{\epsilon_k} \dots \nabla_{\epsilon_1} R_{abc\bar{d}}$, are finite along the members of Γ , where all the components are meant to be measured with respect to a synchronised orthonormal (resp., pseudo-orthonormal) basis field $\{e_{(a)}^a\}$ on \mathcal{U} , in virtue of Theorem 4.2, there must exist a C^{k-} extension $\tilde{\phi} : (\tilde{\mathcal{U}}, \tilde{g}_{ab}|_{\tilde{\mathcal{U}}}) \rightarrow (\tilde{\mathcal{U}}^*, \tilde{g}_{ab}^*)$ of $(\tilde{\mathcal{U}}, \tilde{g}_{ab})$ into $(\tilde{\mathcal{U}}^*, \tilde{g}_{ab}^*)$ so that the images of those members of $\tilde{\Gamma}$ which start on $\tilde{\sigma}_{t_0}$, and which are incomplete and non-extendible in $(\tilde{\mathcal{U}}, \tilde{g}_{ab}|_{\tilde{\mathcal{U}}})$ can be extended, as causal geodesics, in $(\tilde{\mathcal{U}}^*, \tilde{g}_{ab}^*)$.

Consider, now, the selected unique lift $\tilde{\gamma} : (t_1, t_2) \rightarrow \tilde{\mathcal{U}}$ of $\gamma \subset M$, and denote by \mathbf{p} the endpoint of the causal geodesic curve $\tilde{\phi} \circ \tilde{\gamma}$ in $\tilde{\mathcal{U}}^*$ that belongs to the boundary $\partial \tilde{\mathcal{U}}_{[\tilde{\sigma}_{t_0}, \tilde{\epsilon}]} \subset \tilde{\mathcal{U}}^*$. It follows then from the above considerations—in particular, from the facts that $\tilde{\mathcal{U}}$ is a simply connected subset of the universal cover \tilde{M} of M , and also that the metric \tilde{g}_{ab} on \tilde{M} was chosen to be the pull-back of g_{ab} by \mathcal{L}^* —that there has to exist a discrete group $\{\mathbf{i}_m\}$ of isometry actions $\mathbf{i}_m : (\tilde{\mathcal{U}}, \tilde{g}_{ab}|_{\tilde{\mathcal{U}}}) \rightarrow (\tilde{\mathcal{U}}, \tilde{g}_{ab}|_{\tilde{\mathcal{U}}})$, where m takes values from a subset $N = \{1, 2, 3, \dots\}$ of \mathbb{N} , which may be finite or infinite according to whether $\{\mathbf{i}_m\}$ is finitely or infinitely generated. Since, in virtue of Theorem 4.2, the metric \tilde{g}_{ab} extends uniquely to the boundary $\partial \tilde{\mathcal{U}}_{[\tilde{\sigma}_{t_0}, \tilde{\epsilon}]}$ in $\tilde{\mathcal{U}}^*$ the discrete isometry actions $\{\mathbf{i}_m\}$ also extend uniquely to $\partial \tilde{\mathcal{U}}_{[\tilde{\sigma}_{t_0}, \tilde{\epsilon}]}$, i.e., there exists a discrete family $\{\mathbf{i}_m^\#\}$ of isometry actions $\mathbf{i}_m^\# : (\mathcal{U}^\#, \tilde{g}_{ab}^*|_{\mathcal{U}^\#}) \rightarrow (\mathcal{U}^\#, \tilde{g}_{ab}^*|_{\mathcal{U}^\#})$, where $\mathcal{U}^\#$ stands for the closure of $\tilde{\phi}[\tilde{\mathcal{U}}]$ in $\tilde{\mathcal{U}}^*$. Since γ was assumed to terminate on a topological singularity the end point, \mathbf{p} , of $\tilde{\gamma}$ has to be a fixed point with respect to the isometry actions in $\{\mathbf{i}_m^\#\}$. Notice also that—since \mathcal{U} , which according to the above construction is the factor space $\tilde{\mathcal{U}}/\mathcal{C}$, is itself a manifold—all the fixed points of any of the isometry actions in $\{\mathbf{i}_m^\#\}$ have to belong to the boundary $\partial(\tilde{\phi}[\tilde{\mathcal{U}}]) = \mathcal{U}^\# \setminus \text{int}(\mathcal{U}^\#)$ in $\tilde{\mathcal{U}}^*$.

Consider, now, the group $\{L_m\}$ of linear transformations $L_m : T_{\mathbf{p}}(\tilde{\mathcal{U}}^*) \rightarrow T_{\mathbf{p}}(\tilde{\mathcal{U}}^*)$ induced by the members of the discrete isometry group $\{\mathbf{i}_m^\#\}$, i.e., the elements of $\{L_m\}$ are simply the restrictions of the derivatives $\{\mathbf{i}_m^{\#*}\}$ of the maps $\{\mathbf{i}_m^\#\}$ to the tangent space $T_{\mathbf{p}}(\tilde{\mathcal{U}}^*)$ at \mathbf{p} in $\tilde{\mathcal{U}}^*$. Since $\mathbf{i}_m^\#$ are isometry transformations the components of any tensorial object built up from the metric must remain to be intact under the action of $\{L_m\}$ on $T_{\mathbf{p}}(\tilde{\mathcal{U}}^*)$, i.e., their values will be exactly the same regardless whether they are evaluated with respect to a basis $\{e_{(a)}^a\}$ or with respect to any of the bases $\{(L_m(e_{(a)}^a))^a\}$, where $m \in N$. Whence, in

particular, the components \tilde{R}_{abcd} of the Riemann tensor will not be changed either under the action of $\{L_m\}$.

Since the metric \tilde{g}_{ab}^* is Lorentzian the elements of this discrete group $\{L_m\}$ have to be Lorentz transformations. Now, we shall show that $\{L_m\}$ cannot contain a pure rotational subgroup. To see that this has to be the case, assume on the contrary that there is a pure rotational subgroup $\{L_{m_i}\}$ in $\{L_m\}$. Then, in particular, there would exist a timelike vector $t^a \in T_{\mathbf{p}}(\tilde{\mathcal{U}}^*)$ so that t^a would be invariant under the action of the corresponding subgroup $\{L_{m_i}\}$. Consider, now, the future and past inextendible timelike geodesic $\tilde{\lambda}_{\mathbf{p}}$ having t^a as its tangent at \mathbf{p} . Since $\mathbf{p} \in \partial(\tilde{\phi}[\tilde{\mathcal{U}}])$ the timelike geodesic $\tilde{\lambda}_{\mathbf{p}}$ would enter to the interior of $\tilde{\phi}[\tilde{\mathcal{U}}]$ and all the points of $\phi^{-1}[\tilde{\lambda}_{\mathbf{p}}] \cap \tilde{\mathcal{U}}$ were, in fact, fixed points of the associated isometry subgroup $\{i_{m_i}\} \subset \{i_m\}$ acting on $(\tilde{\mathcal{U}}, \tilde{g}_{ab}|_{\tilde{\mathcal{U}}})$. This, however, would lead to the contradictory situation that the points of the curve $\lambda = (\phi^{-1}[\tilde{\lambda}_{\mathbf{p}}] \cap \tilde{\mathcal{U}})/\{i_{m_i}\}$, these were by construction inner points of \mathcal{U} , could not possess open neighbourhoods homeomorphic to open subsets of \mathbb{R}^n . This, in turn, implies that our indirect assumption has to be false, i.e., that $\{L_m\}$ cannot contain a pure rotational subgroup.

As a consequence of the above argument, we have then that each member of the isometry group $\{L_m\}$ must contain a boost constituent, i.e., $\{L_m\}$ cannot be compact, and, in turn, that $\{L_m\}$ is infinitely generated, i.e., $N = \mathbb{N}$ has to hold. This implies then that any subgroup $\{L_{m_i}\}$ of $\{L_m\}$ is so that $\|L_{m_i}\| \rightarrow \infty$, where $\|L_{m_i}\|$ denotes the usual norm of linear maps $L_{m_i} : T_{\mathbf{p}}(\tilde{\mathcal{U}}^*) \rightarrow T_{\mathbf{p}}(\tilde{\mathcal{U}}^*)$, meanwhile the components of the Riemann tensor remain intact under the action of $\{L_{m_i}\}$ on $T_{\mathbf{p}}(\tilde{\mathcal{U}}^*)$. This, however, in virtue of Proposition 6.4.1 of [15], implies then that the geometry of the spacetime $(\tilde{\mathcal{U}}^*, \tilde{g}_{ab}^*)$ has to be specialised at \mathbf{p} .

Summarising the above argument we have then the following.

Proposition 5.1 *Suppose that the conditions of Proposition 3.1 are satisfied, and also that $\gamma : (t_1, t_2) \rightarrow M$ is an incomplete non-extendible causal geodesic in (M, g_{ab}) terminating on a topological singularity. Assume, furthermore, that $t_0 \in (t_1, t_2)$, σ_{t_0} , ε , Γ , $\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}$ and ψ are as they were constructed for the proof of Proposition 3.1. Assume, in addition, that (M, g_{ab}) is globally hyperbolic, and also that the components R_{abcd} of the Riemann tensor, along with the components $\nabla_{e_l} \dots \nabla_{e_1} R_{abcd}$ of its covariant derivatives up to order $0 \leq l \leq k-1$, are uniformly bounded along the members of Γ , and also the line integrals of the components of the k^{th} -order covariant derivatives, $\nabla_{h_k} \dots \nabla_{h_1} R_{abcd}$, are finite along the members of Γ , where all the components are meant to be measured with respect to a synchronised orthonormal (resp., pseudo-orthonormal) basis field $\{e_{(a)}^a\}$ on \mathcal{U} . Let, furthermore, $\tilde{\gamma}$, $\tilde{\Gamma}$, $\tilde{\mathcal{U}}$ and $\tilde{\phi}$ as they were defined above with the help of the covering map \mathcal{C} . Consider the extension $\tilde{\phi} : (\tilde{\mathcal{U}}, \tilde{g}_{ab}|_{\tilde{\mathcal{U}}}) \rightarrow (\tilde{\mathcal{U}}^*, \tilde{g}_{ab}^*)$ of the subspacetime $(\tilde{\mathcal{U}}, \tilde{g}_{ab}|_{\tilde{\mathcal{U}}})$ into a spacetime $(\tilde{\mathcal{U}}^*, \tilde{g}_{ab}^*)$ which is guaranteed to be at least C^{k-} by Theorem 4.2. Then $(\tilde{\mathcal{U}}^*, \tilde{g}_{ab}^*)$ must be specialised at the endpoint $\mathbf{p} \in \tilde{\mathcal{U}}^*$ of the geodesic $\tilde{\gamma} = \tilde{\phi} \circ \gamma$.*

It is worth emphasising that whenever the covering space $(\tilde{\mathcal{U}}^*, \tilde{g}_{ab}^*)$ is special, as it was justified above, at the endpoint $\mathbf{p} \in \tilde{\mathcal{U}}^*$ of the geodesic $\tilde{\gamma} = \tilde{\phi} \circ \gamma$ the original spacetime

(M, g_{ab}) cannot be generic either since its Riemann tensor has to become more and more special while approaching the ‘ideal endpoint’ of γ , i.e., as $t \rightarrow t_2$.

6 The global extension

This section is to provide the desired global extension based on the use of our intermediate extension $\phi : (\mathcal{U}, g_{ab}|_{\mathcal{U}}) \rightarrow (\mathcal{U}^*, g_{ab}^*)$. To start off consider a spacetime (M, g_{ab}) containing a geodesically incomplete non-extendible timelike (resp., null) geodesic $\gamma : (t_1, t_2) \rightarrow M$ which does not terminate either on a tidal force tensor singularity or on a topological singularity. Let, furthermore, $t_0 \in (t_1, t_2)$, σ_{t_0} , the $n - 1$ -parameter family of timelike (resp., null) geodesics Γ and $\mathcal{U}_{[\sigma_{t_0}, \varepsilon]}$ as they were defined in section 3 so that Gaussian (resp., Gaussian null) coordinates are globally well-defined on \mathcal{U} . Assume, in addition, that the spacetime (M, g_{ab}) is globally hyperbolic, and also that the components R_{abcd} of the Riemann tensor, along with the components $\nabla_{\epsilon_1} \dots \nabla_{\epsilon_l} R_{abcd}$ of its covariant derivatives up to order $0 \leq l \leq k - 1$ are uniformly bounded along the members of Γ , and also the line integrals of the components of the k^{th} -order covariant derivatives, $\nabla_{h_k} \dots \nabla_{h_1} R_{abcd}$, are finite along the members of Γ , where all the components are meant to be measured with respect to a synchronised orthonormal (resp., pseudo-orthonormal) basis field $\{e_{(a)}^a\}$ on \mathcal{U} . Then, in virtue of Theorem 4.2, there exists a C^{k-} extension $\phi : (\mathcal{U}, g_{ab}|_{\mathcal{U}}) \rightarrow (\mathcal{U}^*, g_{ab}^*)$ of the subspacetime $(\mathcal{U}, g_{ab}|_{\mathcal{U}})$ into a spacetime $(\mathcal{U}^*, g_{ab}^*)$ so that the images of those members of Γ which start on σ_{t_0} and which are incomplete and non-extendible in M can be extended, as timelike (resp., null) geodesics, in $(\mathcal{U}^*, g_{ab}^*)$. Our aim in this section is to show that since the spacetime (M, g_{ab}) is globally hyperbolic a global extension $(\widehat{M}, \widehat{g}_{ab})$ of (M, g_{ab}) can be given, by making use of this isometric imbedding $\phi : (\mathcal{U}, g_{ab}|_{\mathcal{U}}) \rightarrow (\mathcal{U}^*, g_{ab}^*)$.

The desired global extension will be performed by gluing the spacetimes (M, g_{ab}) and $(\mathcal{U}^*, g_{ab}^*)$ together with the help of the intermediate extension $\phi : (\mathcal{U}, g_{ab}|_{\mathcal{U}}) \rightarrow (\mathcal{U}^*, g_{ab}^*)$. More precisely, the enlarged spacetime manifold \widehat{M} is defined as follows. Notice first that with the help of the isometric imbedding $\phi : (\mathcal{U}, g_{ab}|_{\mathcal{U}}) \rightarrow (\mathcal{U}^*, g_{ab}^*)$ we may define an equivalence relation \mathcal{R} on the union of M and \mathcal{U}^* by requiring that two points p and p^* of the union $M \cup \mathcal{U}^*$ to be equivalent if $p \in M$ and $p^* \in \mathcal{U}^*$, and $\phi(p) = p^*$. Now, the base manifold \widehat{M} of the enlarged spacetime is defined to be the factor space

$$\widehat{M} = (M \cup \mathcal{U}^*) / \mathcal{R}. \quad (6.1)$$

Since both M and \mathcal{U}^* are smooth n -dimensional differentiable manifolds, and since the equivalence relation \mathcal{R} is defined with the help of the map ϕ which is a smooth diffeomorphism between \mathcal{U} and $\phi[\mathcal{U}] \subset \mathcal{U}^*$ the factor space \widehat{M} necessarily possesses the structure of a smooth manifold (see also Lemma 4.1 of [43]). However, as Example 6.1 below indicates, unless the boundary of \mathcal{U} in M is guaranteed to be trivial the factor space \widehat{M} may not be a Hausdorff manifold. Before proceeding, recall that the topology of the manifold $\widehat{M} = (M \cup \mathcal{U}^*) / \mathcal{R}$, which necessarily is the ‘factor topology’ on \widehat{M} , is always uniquely determined by the topology of M and \mathcal{U}^* , along with the equivalence relation \mathcal{R} . In particular,

it is said that $\widehat{\mathcal{O}}$ is an open subset in \widehat{M} if its pre-image $\Pi^{-1}[\widehat{\mathcal{O}}]$ is open in $M \cup \mathcal{U}^*$, where $\Pi : M \cup \mathcal{U}^* \rightarrow \widehat{M}$ denotes the projection of $M \cup \mathcal{U}^*$ into $(M \cup \mathcal{U}^*)/\mathcal{R}$, mapping each point $p^\# \in M \cup \mathcal{U}^*$ to the equivalence class $[p^\#] \in \widehat{M}$. Moreover, the open sets of $M \cup \mathcal{U}^*$ are uniquely determined by the union of the open subsets of M and \mathcal{U}^* , respectively.

The following simple example makes it transparent that, in general, without imposing a restriction on the causal structure of the spacetime, the boundary $\partial\mathcal{U}$ of \mathcal{U} is not guaranteed to be simple¹³ and—what is even more inconvenient from our point of view—that the topology of the factor space $\widehat{M} = (M \cup \mathcal{U}^*)/\mathcal{R}$ may not be Hausdorff.

Example 6.1 Choose (M, η_{ab}) to be the subspacetime of the three-dimensional Minkowski spacetime, $(\mathbb{R}^3, \eta_{ab})$, with Cartesian coordinates (x, y, t) , from which the spacelike line segment

$$\lambda = \{(x, y, t) \in \mathbb{R}^3 \mid y = t = 0 \text{ and } x \in [-\delta, \delta]\} \quad (6.2)$$

where δ is a positive number, is removed, i.e., $M = \mathbb{R}^3 \setminus \lambda$. Let, furthermore, the timelike geodesic curve γ and the subset σ_{t_0} of the spacelike surface Σ , given as $t = -1$, be chosen as

$$\gamma = \{(x, y, t) \in \mathbb{R}^3 \mid x = y = 0 \text{ and } t < 0\} \quad (6.3)$$

and

$$\sigma_{t_0} = \{(x, y, t) \in \mathbb{R}^3 \mid x, y \in (-2\delta, 2\delta) \text{ and } t_0 = -1\}. \quad (6.4)$$

Then, by following the main steps of the general construction applied in the previous sections it is straightforward to see that for any particular value of $\delta > 0$ and $\varepsilon > 0$ the subsets \mathcal{U}^* and \mathcal{U} can be given as

$$\mathcal{U}^* = (-2\delta, 2\delta) \times (-2\delta, 2\delta) \times (-1, \varepsilon) \subset \mathbb{R}_*^3 \quad (6.5)$$

and

$$\mathcal{U} = \mathcal{U}^* \setminus [-\delta, \delta] \times \{0\} \times [0, \varepsilon) \subset \mathbb{R}^3, \quad (6.6)$$

where the Cartesian product $[-\delta, \delta] \times \{0\} \times [0, \varepsilon)$ is nothing but the “shadow”, \mathcal{S}_λ , of λ in \mathcal{U}^* , generated by timelike geodesics starting with tangent vector $t^\alpha = (0, 0, 1)$ at the points of λ in the original three-dimensional Minkowski spacetime. (Notice that the lowercase star “ $*$ ” is used only to distinguish two copies of \mathbb{R}^3 or \mathcal{S}_λ in the present example.) It follows then that $\mathcal{S}_\lambda \setminus [-\delta, \delta] \times \{0\} \times \{0\}$ is a proper subset of M , and also that $\mathcal{S}_{\lambda*}$ is a proper subset of \mathcal{U}^* , while \mathcal{S}_λ does belong to the complement of \mathcal{U} , i.e., $\mathcal{S}_\lambda \not\subset \mathcal{U}$. Since, in the present case the map ϕ is nothing but the restriction of the “identity” map—identifying the points of the two copies of \mathbb{R}^3 labelled by the same 3-tuples—to \mathcal{U} we have that $\mathcal{S}_{\lambda*} \not\subset \phi[\mathcal{U}]$. Thereby, the equivalence relation \mathcal{R} determined by ϕ does not identify the pair of points $p \in \mathcal{S}_\lambda \subset M$ and $p^* \in \mathcal{S}_{\lambda*} \subset \mathcal{U}^*$ which, on the other hand, do possess the same coordinates in \mathbb{R}^3 and \mathbb{R}_*^3 , respectively. Notice, however, that the points in sufficiently small open neighbourhoods of p and p^* in M and \mathcal{U}^* on both sides of \mathcal{S}_λ and $\mathcal{S}_{\lambda*}$ having the same coordinates in \mathbb{R}^3

¹³Here $\partial\mathcal{U}$ is considered to be simple if any point of the closure $\overline{\mathcal{U}}$ of \mathcal{U} , in M , can be reached along a unique member of Γ starting at a point of the closure $\overline{\sigma_{t_0}} \subset \Sigma$.

and \mathbb{R}_*^3 , respectively, will be identified. By making use of the above recalled definition of the factor topology it is straightforward to justify then that the pair of points $[p]$ and $[p^*]$ cannot be separated by open neighbourhoods in $\widehat{M} = (M \cup \mathcal{U}^*)/\mathcal{R}$, i.e., the topology of \widehat{M} is not Hausdorff.

In returning to the general case recall first that each point of \mathcal{U} can be reached, by following one of the uniquely determined member of the $n - 1$ -parameter causal geodesic congruence Γ , from a point of σ_{t_0} , and also that σ_{t_0} was chosen to have compact closure in Σ . By making use of the definition, (6.1), of \widehat{M} it is straightforward to see that the factor topology of \widehat{M} is guaranteed to be Hausdorff if the boundary $\partial\mathcal{U}$ of \mathcal{U} in M is “simple”, i.e., whenever any point of the closure $\overline{\mathcal{U}}$ of \mathcal{U} , in M , can be reached along a unique member of Γ starting at a point of the closure $\overline{\sigma_{t_0}} \subset \Sigma$. The following proposition is to show that the boundary of \mathcal{U} in M is guaranteed to be simple whenever (M, g_{ab}) is globally hyperbolic.

Proposition 6.1 *Suppose that the conditions of Theorem 4.2 hold, in particular, that σ_{t_0} and \mathcal{U} are chosen accordingly, and also that (M, g_{ab}) is globally hyperbolic. Then, to any point $q \in \partial\mathcal{U}$ there exists a unique causal geodesic $\gamma_q \in \Gamma$ through q so that γ_q intersects Σ at some point of $\overline{\sigma_{t_0}}$.*

Proof: Assume, on contrary to our claim, that there exists $q \in \partial\mathcal{U}$ so that q cannot be achieved along any of the members of Γ through the points of $\overline{\sigma_{t_0}}$. Since, by construction $\sigma_{t_0} \subset \partial\mathcal{U}$ we may assume, without loss of generality, that $q \in \partial\mathcal{U} \setminus \overline{\sigma_{t_0}}$

Since \mathcal{U} is an open subset of M and $q \in \partial\mathcal{U}$, there has to exist a point sequence $\{q_i\}$ in \mathcal{U} so that q is a limit point of $\{q_i\}$. Consider, then, the sequence of causal geodesics $\{\gamma_i\}$ comprised by the unique members of Γ through the points q_i . Since (M, g_{ab}) is a smooth spacetime, without loss of generality, the causal geodesics $\{\gamma_i\}$ will be assumed to be future and past inextendible in (M, g_{ab}) . Since $\{q_i\} \subset \mathcal{U}$ each member of the sequence $\{\gamma_i\}$ intersects σ_{t_0} at certain point $\pi(q_i) = \gamma_i \cap \sigma_{t_0}$. Furthermore, because the closure $\overline{\sigma_{t_0}}$ of σ_{t_0} is a compact subset of Σ any limit point of the sequence $\{\pi(q_i)\}$ should also belong to $\overline{\sigma_{t_0}}$. Denote by $\pi(q)$ one of these limit points.

Now, in virtue of Lemma 6.2.1 of [29], there must exist future and past inextendible non-spacelike curves $\gamma_{\pi(q)}$ and γ_q to the sequence $\{\gamma_i\}$ so that both $\gamma_{\pi(q)}$ and γ_q are the limit curves of the sequence $\{\gamma_i\}$ through the points of $\pi(q) \in \overline{\sigma_{t_0}}$ and $q \in \partial\mathcal{U}$, respectively. Notice that by construction $\gamma_{\pi(q)}$ has to coincide with the member of Γ through the point $\pi(q)$. Consider now a Cauchy surface \mathcal{C} of (M, g_{ab}) through $\pi(q) \in \overline{\sigma_{t_0}}$. Our aim is to show now that γ_q cannot intersect \mathcal{C} which, in turn, contradicts the assumption that (M, g_{ab}) is a globally hyperbolic spacetime since each of the future and past inextendible causal curves in (M, g_{ab}) are required to meet any Cauchy surface precisely once [26].

Assume that the future and past inextendible causal curve γ_q intersects \mathcal{C} , say at a point $q' = \gamma_q \cap \mathcal{C}$. According to the above construction this point is necessarily different from $\pi(q)$. Notice that, since the entire of γ_q , including the point q' , is required to be a limit curve of the sequence $\{\gamma_i\}$ to any arbitrarily small open neighbourhood $\mathcal{O}_{q'}$ of q' there exists

a causal geodesic in the sequence $\{\gamma_i\}$ that intersects \mathcal{C} in $\mathcal{O}_{q'}$. The existence of such a causal curve is, however, excluded by the assumption that \mathcal{C} is a Cauchy surface of the globally hyperbolic spacetime (M, g_{ab}) since the corresponding member of $\{\gamma_i\}$ would intersect \mathcal{C} twice. Consequently, in either case, we get in conflict with the assumption that (M, g_{ab}) is globally hyperbolic which contradiction can only be resolved if our indirect hypotheses is false, i.e., the boundary $\partial\mathcal{U}$ of \mathcal{U} has to be simple. \square

We would like to emphasise that the condition, requiring the spacetime to be globally hyperbolic, of the above proposition might not be optimal, i.e., the above assertion might also be valid for spacetimes with less regular causal structure. What is justified by the above argument is that whenever the spacetime (M, g_{ab}) is globally hyperbolic the boundary $\partial\mathcal{U}$ of \mathcal{U} , in M , is necessarily trivial.

Notice also that the main part of the argument of Proposition 4.1, along with that of Theorem 4.2, Theorem 5.1 and that of the above proposition, would remain intact if instead of requiring the entire spacetime (M, g_{ab}) to be globally hyperbolic only the existence of a globally hyperbolic subspacetime $(M', g_{ab}|_{M'})$ was assumed for which $\overline{\mathcal{U}} \subset M'$ holds. If such a globally hyperbolic subspacetime $(M', g_{ab}|_{M'})$ existed one could refer to the underlying spacetime (M, g_{ab}) as being ‘locally globally hyperbolic’ in a neighbourhood of a final segment of the incomplete and non-extendible timelike (resp., null) geodesic γ .

Now we are in the position to provide the global extension $\Phi : (M, g_{ab}) \rightarrow (\widehat{M}, \widehat{g}_{ab})$ as follows. According to the above argument \widehat{M} is guaranteed to be a smooth Hausdorff differentiable manifold. Moreover, since \widehat{M} was chosen to be the factor space $\widehat{M} = (M \cup \mathcal{U}^*)/\mathcal{R}$ an imbedding Φ of M into \widehat{M} can be given as

$$\Phi = \Pi \circ \begin{cases} \phi, & \text{on } \mathcal{U}; \\ \text{identity}, & \text{elsewhere,} \end{cases} \quad (6.7)$$

where Π denotes, as above, the natural projection from $M \cup \mathcal{U}^*$ into \widehat{M} . Since $\phi : (\mathcal{U}, g_{ab}|_{\mathcal{U}}) \rightarrow (\mathcal{U}^*, g_{ab}^*)$ is guaranteed to be a C^{k-} extension of $(\mathcal{U}, g_{ab}|_{\mathcal{U}})$ into $(\mathcal{U}^*, g_{ab}^*)$ it is straightforward to see that the metrics g_{ab} and g_{ab}^* on M and \mathcal{U}^* , respectively, uniquely determine a C^{k-} metric \widehat{g}_{ab} on \widehat{M} , and also that Φ gets to be an isometric imbedding of (M, g_{ab}) into $(\widehat{M}, \widehat{g}_{ab})$, i.e., $\Phi^* g_{ab} = \widehat{g}_{ab}|_{\Phi[M]}$.

The combination of all the above results, in particular, that of the above argument, along with the assertions of Theorem 4.2, Proposition 5.1 and Proposition 6.1, yields the proof of the following.

Theorem 6.1 *Assume that (M, g_{ab}) is a generic globally hyperbolic causal geodesically incomplete spacetime. Let $\gamma : (t_1, t_2) \rightarrow M$ be one of the incomplete non-extendible timelike (resp., null) geodesic curves in M , which does not terminate on a tidal force tensor singularity, and which, due to the genericness of (M, g_{ab}) , does not terminate on a topological singularity either. Let, furthermore, $t_0 \in (t_1, t_2)$, $\sigma_{t_0} \subset \Sigma$, Γ , \mathcal{U} and a synchronised orthonormal (resp., pseudo-orthonormal) basis field $\{e_{(a)}^a\}$ on \mathcal{U} as they were selected in section 3. Suppose, finally, that the components R_{abcd} of the Riemann tensor, along with the*

components $\nabla_{\epsilon_1} \dots \nabla_{\epsilon_l} R_{abcd}$ of its covariant derivatives up to order $0 \leq l \leq k-1$ are bounded on \mathcal{U} , and also the line integrals of the components of the k^{th} -order covariant derivatives, $\nabla_{h_k} \dots \nabla_{h_1} R_{abcd}$, are finite along the members of Γ , where all the components are meant to be measured with respect to a synchronised orthonormal (resp., pseudo-orthonormal) basis field $\{e_{(a)}^a\}$ on \mathcal{U} . Then there exists a global C^{k-} extension $\Phi : (M, g_{ab}) \rightarrow (\widehat{M}, \widehat{g}_{ab})$ of the spacetime (M, g_{ab}) so that for each member $\bar{\gamma}$ of the congruence Γ which intersects Σ in σ_{t_0} and which is incomplete and non-extendible in (M, g_{ab}) the timelike (resp., null) geodesic curve $\Phi \circ \bar{\gamma}$ is extendible in $(\widehat{M}, \widehat{g}_{ab})$.

The following example is to demonstrate that to have a global extension of the above type to a causal geodesically incomplete spacetime it is simply not sufficient to refer to the behaviour of the Riemann tensor along a single incomplete causal geodesic. Rather, we always have to use the information associated with an $n-1$ -parameter family of synchronised causal geodesics covering an open neighbourhood of a final segment of the selected one.

Example 6.2 Start with the two-dimensional Minkowski spacetime $(\mathbb{R}^2, \eta_{ab})$ with the Cartesian coordinates (x, t) , and denote by $J^+(o)$ the causal future of the origin. Choose M to be $\mathbb{R}^2 \setminus J^+(o)$. Let, now, Ω be a smooth function which is defined on M such that it takes the value 1 except on the region $0 < t < -x$, where it is chosen so that the scalar curvature of the spacetime $(M, \Omega^2 \eta_{ab})$ blows up everywhere on the part of the boundary given as $t = -x > 0$. Notice that regardless of the actual form of Ω this construction guarantees that the curvature remains identically zero on the other part of the boundary $t = x > 0$. A particular choice for such a function, Ω , can be given as

$$\Omega = \begin{cases} e^{-\frac{t^2}{x^2-t^2}}, & \text{if } 0 < t < -x; \\ 1, & \text{otherwise.} \end{cases} \quad (6.8)$$

Clearly, the spacetime $(M, \Omega^2 \eta_{ab})$ is globally hyperbolic since only a future set has been removed from the globally hyperbolic two-dimensional Minkowski spacetime. Consider, now, the one-parameter family of timelike geodesics $\{\gamma_{\delta'}\}$ starting at the points of $\Sigma = \{(x, t) \mid t = -1, x \in (-\delta, \delta)\}$, where δ is a positive number, with tangent t^a the components of which are $(0, 1)$. Clearly all of the geodesics $\gamma_{\delta'}$ with $\delta' \in [0, \delta)$ are straight lines hitting the boundary $t = x \geq 0$ without experiencing any sort of tidal effects. On the other hand, the timelike geodesics $\gamma_{\delta'}$ with $\delta' \in (-\delta, 0)$ after entering to the region $0 < t < -x$, where Ω is not identically 1, they start to bend to the right towards the boundary and the scalar curvature, along with the “tidal force tensor” component R_{1212} , blows up along them. Clearly, by starting with any of the timelike geodesics $\gamma_{\delta'}$ with $\delta' \in (0, \delta)$ and with a sufficiently small neighbourhood σ_{t_0} of $\gamma_{\delta'} \cap \Sigma$ in Σ a global extension of the spacetime $(M, \Omega^2 \eta_{ab})$ can be given, while no such extension can be based on any of the timelike geodesics $\gamma_{\delta'}$ with $\delta' \in (-\delta, 0)$. Notice that along $\gamma_{\delta'=0}$, separating the above two subfamilies, although the “tidal force tensor” component R_{1212} remains identically zero, regardless how small the neighbourhood σ_{t_0} was chosen, it is not possible to provide a global extension of $(M, \Omega^2 \eta_{ab})$ centred on $\gamma_{\delta'=0}$. Notice, however, that in accordance with the main result of [42], it is straightforward to perform a

local extension of $(M, \Omega^2 \eta_{ab})$ so that the image of $\gamma_{\delta'=0}$ can be extended in the corresponding local extension.

7 Final remarks

Our aim in this paper was to identify the sufficient conditions ensuring the existence of a global extension to a *generic* causal geodesically incomplete spacetime. According to the presented results, whenever a (future) incomplete inextendible timelike (resp., null) geodesic curve does not terminate on a tidal force tensor singularity, by making use of an $n - 1$ -parameter family of synchronised timelike (resp., null) geodesics Γ Gaussian (resp., Gaussian null) coordinates can be defined in a neighbourhood \mathcal{U} of a final segment of the selected incomplete inextendible timelike (resp., null) geodesic curve. It was also shown that if, in addition, the spacetime, (M, g_{ab}) , is globally hyperbolic, and the components of the Riemann tensor, and that of its covariant derivatives up to order $k - 1$ are bounded on \mathcal{U} , and also the line integrals of the components of the k^{th} -order covariant derivatives are finite along the members of Γ , where all the components are meant to be measured with respect to a synchronised orthonormal (resp., pseudo-orthonormal) basis field on \mathcal{U} , then there exists a C^{k-} extension $\Phi : (M, g_{ab}) \rightarrow (\widehat{M}, \widehat{g}_{ab})$ so that for each member $\bar{\gamma} \in \Gamma$, which is (future) inextendible in (M, g_{ab}) , the geodesic, $\Phi \circ \bar{\gamma}$, is extendible in $(\widehat{M}, \widehat{g}_{ab})$.

It is important to keep in mind that Theorem 6.1, formulating the above assertions, is essentially an *existence theorem*. Nevertheless, since the existence of the global extension $\Phi : (M, g_{ab}) \rightarrow (\widehat{M}, \widehat{g}_{ab})$ was demonstrated by explicitly performing several succeeding constructive steps, we believe that the associated constructions could also be useful in performing global extensions of various particular causal geodesically incomplete spacetimes satisfying the above requirements. Notice also that the scope of our investigations were limited in the sense that considerations were restricted exclusively to the extendibility of the differentiable and metric structures of spacetimes. In particular, the extendibility of other physical fields, such as various possible matter fields which could also be present on M , were left out from the presented considerations. Nevertheless, it seems to be plausible that many of the techniques developed and applied here, for the study of the behaviour of the coordinate components of the metric, should also be applicable in case of the analogous investigations of the coordinate components of certain tensor fields, representing the matter content of a spacetime, whenever they satisfy, say, suitable hyperbolic evolution equations. Clearly, the study of the related issues would be of obvious importance.

Spacetimes with a topological singularity which otherwise are regular, in the sense that they do satisfy all the other conditions guaranteeing the existence of a global extension, were also investigated. It was demonstrated that to such a spacetime or, more precisely, to an appropriate subspacetime of such a spacetime, it is always possible to find a covering that can be extended. Noticing then that there must exist a group of discrete isometry actions on the covering spacetime it was possible to prove that this covering space cannot be generic, which, in turn, implies that the original spacetime cannot be generic either. We would like to emphasise that this result strongly supports the expectation that there may not be other

way of “producing” spacetimes with a topological singularity except those procedures that have yielded the known particular examples (see, e.g., Refs. [18, 11] for a good collection of these type of examples). It would be important to know whether with the help of suitable hyperbolic field equations even the existence of a (possibly local) one-parameter group of isometries could also be confirmed. In trying to justify such a conjecture one might start, e.g., with the coupled Einstein matter field equations and try to apply a suitable adaptation of the techniques used in [20, 44, 45, 46, 47] to prove the existence of a one-parameter family of spacetime symmetries. The investigation of this problem would definitely deserve further attention.

It is worth keeping in mind that the extension $(\widehat{M}, \widehat{g}_{ab})$ of the spacetime (M, g_{ab}) by no means is guaranteed to be maximal. The enlarged spacetime $(\widehat{M}, \widehat{g}_{ab})$ provides merely a global extension through a “locally regular part of the boundary” of an incomplete but otherwise arbitrary spacetime (M, g_{ab}) . In this respect $(\widehat{M}, \widehat{g}_{ab})$ could be called—by making use of an apparently contradictory terminology—to be a “local global extension”. Notice, however, that by making use of the notion of isometric imbedding a partial order on the set of spacetimes can be introduced. Then, by making use of an argument of the type that had been applied by Choquet-Bruhat and Geroch [10] to show the existence of a unique maximal Cauchy development—or by the more sophisticated argument of Clarke [12] based on the use of partially ordered nets of equivalence classes of spacetime models—it might be possible to justify the existence of a truly global extension to any given particular extendible spacetime.

We would like to emphasise again that the main motivation for the investigation of spacetime extensions have been provided by the implications of the singularity theorems some of which were proved more than forty years ago [38, 30]. In this respect it is also worth recalling that the singularity theorems are frequently referred in providing evidences for the necessity that general relativity has to be quantised. The corresponding arguments assume implicitly that while approaching to a singularity the curvature gets to be unboundedly large thereby general relativity cannot be used there. Actually, this type of reasoning and the associated belief is part of the ‘scientific folklore’ despite the fact that the singularity theorems of Penrose and Hawking do predict merely the existence of incomplete causal geodesics in a wide class of physically plausible spacetimes describing the expanding universe and the gravitational collapse of stars. More concretely, there is no such a universal argument that would guarantee that the curvature should necessarily blow up, say, along any of the predicted incomplete geodesics. Interestingly, in spite of the presence of the corresponding gap in the ‘classical’ argument, in some of the recent investigations carried out in a mini-superspace version of loop quantum gravity, applied to the simplest isotropic cosmological and spherically symmetric black hole configurations, it is claimed in the relevant investigations [6, 7, 1] that loop quantum gravity resolves the problems related to existence of spacetime singularities.

In consequence of the above mentioned gap in the classical argument there is a clearly manifested and long lasting desire to acquire more knowledge about spacetime singularities. Since all the attempts have been failed in providing a general argument telling us what

exactly goes wrong along an incomplete causal geodesic various type of approaches have been developed which aim to characterise the singular behaviour of specific class of causal geodesically incomplete spacetimes. As a pay back for giving up generality more detailed study of the singular behaviour of various field variables of the explicit spacetime models were possible to be done. Impressive investigations of this type, where generality is tried to be preserved up to certain extent, and, in addition, the concept of “generalised hyperbolicity” were introduced, can be found, e.g, in Refs. [16, 59, 60]. In the corresponding considerations singularities are regarded as obstructions to the Cauchy development of physical fields rather than as obstructions to extendibility of causal geodesics. The realisation of this approach, however, requires a much more sophisticated theory of generalised functions than the usual distributions [59, 52], therefore, it is hard to foresee whether the proposed approach may turn to be viable.

Completely independent motivations—aiming also the justification of the strong cosmic censor hypothesis (following the conceptual approach proposed by Eardley and Moncrief in [17])—led to the detailed investigations of particular cosmological models. Immediate examples of this type are, e.g., the investigations of the oscillatory behaviour of singularities in Bianchi type VIII or IX models [50, 51] or the investigation of the oscillatory behaviour of the Gowdy spacetimes, with various symmetry assumptions, close to their singularities [3, 4, 27, 32]. This family can also be successfully investigated with analytic methods based on the use of Fuchsian-type of evolution equations [48, 49, 34]. In spite of the unquestionable advances of these investigations in providing detailed information about the way certain quantities get to be singular while approaching the singularity they are also limited in the sense that they are suitable to investigate only very limited classes of explicit spacetime models, whereas, for instance, the results covered by the present paper may immediately be applied without limitation to either of the spacetimes which, in addition, to the assumptions we have made does also satisfy the conditions of one of the singularity theorems.

As a consequence of the above discussions it is of obvious interest to know whether the presented extension results may have any implications in connection with the singularity theorems. For the first glance it might seem to be plausible to assume that our global extension result is hardly useful in this respect since throughout the above outlined investigations, the original spacetime was assumed to be smooth and, under suitable conditions, the extension was guaranteed to be C^{k-} . Therefore, it is of principal importance to know to what extent the associated constructive elements of the applied extension procedure can also be performed in case of spacetimes belonging to lower differentiability classes. It turned out that a sensible generalisation can be provided for the class of spacetimes in which the Einstein’s equations may only be defined as distributions. More importantly, it was found that the relevant global extension results make it also possible to strengthen the conclusion of the singularity theorems of Penrose and Hawking so that not only the existence of incomplete causal geodesics can be predicted but, in addition, the blowing up of certain components of the curvature tensor along some of the infinitesimally close causal geodesics can also be demonstrated in generic globally hyperbolic spacetimes. The corresponding generalisation of our global extension result, along with the investigation of its relevance in connection

with the singularity theorems, will be published elsewhere.

Acknowledgements

The author is grateful to Akihiro Ishibashi and Christian Lübbe for their helpful comments on a former version of this paper. This research was supported in parts by OTKA grant K67942.

References

- [1] A. Ashtekar and M. Bojowald: *Quantum geometry and the Schwarzschild singularity*, Class. Quant. Grav. **23**, 391-411 (2006)
- [2] J.K. Beem, P.E. Ehrlich and K.L. Easley: *Global Lorentzian geometry*, 2nd ed., Marcel Dekker Inc. New York (1996)
- [3] B.K. Berger and V. Moncrief: *Evidence for an oscillatory singularity in generic $U(1)$ symmetric cosmologies on $T^3 \times \mathbb{R}$* , Phys. Rev. D. **58**, 064023 (1998)
- [4] B.K. Berger, J. Isenberg and M. Weaver: *Oscillatory approach to the singularity in vacuum spacetimes with T^2 isometry*, Phys. Rev. D. **64**, 084006 (2001)
- [5] A.N. Bernal and M. Sánchez: *Globally hyperbolic spacetimes can be defined as "causal" instead of "strongly causal"*, Class. Quant. Grav. **24**, 745-750 (2007)
- [6] M. Bojowald: *Homogeneous loop quantum cosmology*, Class. Quant. Grav. **20**, 2595-2615 (2003)
- [7] M. Bojowald: *Loop quantum cosmology*, Living Reviews in Relativity (2005)
<http://relativity.livingreviews.org/Articles/lrr-2005-11/>
- [8] R.H. Boyer and R.W. Lindquist: *Maximal analytic extension of the Kerr metric*, J. Math. Phys. **8**, 265-281 (1967)
- [9] B. Carter: *Global structure of the Kerr family of gravitational field*, Phys. Rev. **174**, 1559-1571 (1968)
- [10] Y. Choquet-Bruhat and R. Geroch: *Global aspects of the Cauchy problem in general relativity*, Commun. Math. Phys. **14**, 329-335 (1969)
- [11] C.J.S. Clarke: *Singularities in globally hyperbolic space-time*, Commun. Math. Phys. **41**, 65-78 (1975)
- [12] C.J.S. Clarke: *Space-time singularities*, Commun. Math. Phys. **49**, 17-23 (1976)
- [13] C.J.S. Clarke: *Local extensions in singular space-times II*, Commun. Math. Phys. **84**, 329-331 (1982)
- [14] C.J.S. Clarke: *Space-times of low differentiability and singularities*, J. Math. Anal. Appl. **88**, 270-305 (1982)
- [15] C.J.S. Clarke: *The analysis of space-time singularities*, (Cambridge lecture notes in physics, 1), Cambridge, Univ. Press (1993)

- [16] C.J.S. Clarke: *Generalised hyperbolicity in singular space-times*, Class. Quant. Grav. **15**, 975-984 (1998)
- [17] D. Eardley and V. Moncrief: *The global existence problem and cosmic censorship in general relativity*, Gen. Rel. Grav. **13**, 887-892 (1981)
- [18] G.F.R. Ellis and B.G.Schmidt: *Singular space-times*, Gen. Rel. Grav. **8**, 915-953 (1977)
- [19] G.F.R. Ellis and B.G.Schmidt: *Classification of singular space-times*, Gen. Rel. Grav. **10**, 989-997 (1979)
- [20] H. Friedrich, I. Rácz and R.M. Wald: *On the rigidity theorem for spacetimes with a stationary event horizon or a compact Cauchy horizon*, Commun. Math. Phys. **204** 691-707 (1999)
- [21] R. Geroch: *Singularities in closed universes*, Phys. Rev. Lett. **17**, 445-447 (1966)
- [22] R. Geroch: *Topology in general relativity*, J. Math. Phys. **8**, 782-786 (1967)
- [23] R. Geroch: *Local characterisation of singularities in general relativity*, J. Math. Phys. **9**, 450-465 (1968)
- [24] R. Geroch: *What is a singularity in general relativity?*, Annals Phys. **48**, 526-540 (1968)
- [25] R. Geroch: *Singularities*, in *Relativity*, Eds. M. Carmeli, S.I. Fickler and L. Witten, Plenum Press, New York, p. 259-291 (1970)
- [26] R. Geroch: *Domain of dependence*, J. Math. Phys. **11**, 437-449 (1970)
- [27] B. Grubišić and V. Moncrief: *Asymptotic behaviour of the $T^3 \times \mathbb{R}$ Gowdy spacetimes*, Phys. Rev. D. **47**, 2371-2382 (1993)
- [28] S.G. Harris: *A triangle comparison theorem for Lorentzian manifolds*, Indiana Math. J., **31**, 289-308 (1982)
- [29] S.W. Hawking and G.R.F. Ellis: *The large scale structure of spacetime*, Cambridge University Press, Cambridge, 1973)
- [30] S.W. Hawking: *The occurrence of singularities in cosmology. III. Causality and singularities*, Proc. Roy. Soc. Lond. Ser. A **300**, 182-2001 (1967)
- [31] S.W. Hawking and R. Penrose: *The singularities of gravitational collapse and cosmology*, Proc. Roy. Soc. Lond. Ser. A **314**, 529-548 (1970)
- [32] J. Isenberg and S. Kichenassamy: *Asymptotic behaviour in polarised T^2 -symmetric vacuum spacetimes*, J. Math. Phys. **40**, 340-352 (1999)
- [33] R. Johnson: *The bundle boundary in some special cases*, J. Math. Phys. **18**, 898-902 (1977)
- [34] S. Kichenassamy and A.D. Rendall: *Analytic description of singularities in Gowdy spacetimes*, Class. Quant. Grav. **15**, 1339-1355 (1998)
- [35] M.D. Kruskal: *Maximal extension of the Schwarzschild metric*, Phys. Rev. **119**, 1743-1745 (1960)
- [36] E. Minguzzi: *Limit curve theorems in Lorentzian geometry*, J. Math. Phys. **49**, 092501 (2008)

- [37] B. O'Neill: *Semi-Riemannian geometry*, Academic Press (1983)
- [38] R. Penrose: *Gravitational collapse and space-time singularities*, Phys. Rev. Lett. **14** 54-59 (1965)
- [39] R. Penrose: *Gravitational collapse: The role of general relativity*, Riv. del Nuovo Cimento **1** (Numero Spec.) 256-275 (1969)
- [40] R. Penrose: *Techniques of differential topology in relativity*, SIAM, No. 7., Philadelphia (1972)
- [41] I. Rácz and R. M. Wald: *Extension of spacetimes with Killing horizon*, Class. Quant. Grav. **9**, 2643-2656 (1992)
- [42] I. Rácz: *Space-time extensions I*, J. Math. Phys. **34**, 2448-2464 (1993)
- [43] I. Rácz and R.M. Wald: *Global extension of spacetimes describing asymptotic final states of black holes*, Class. Quant. Grav. **13**, 539-553 (1996)
- [44] I. Rácz: *On further generalisation of the rigidity theorem for spacetimes with a stationary event horizon or a compact Cauchy horizon*, Class. Quant. Grav. **17** 153-178 (2000)
- [45] I. Rácz: *On the existence of Killing vector fields*, Class. Quant. Grav. **16**, 1695-1703 (1999)
- [46] I. Rácz: *Symmetries of spacetime and their relation to initial value problems*, Class. Quant. Grav. **18**, 5103-5113 (2001)
- [47] I. Rácz: *Stationary black holes as holographs*, Class. Quant. Grav. **24**, 5541-5571 (2007)
- [48] A.D. Rendall: *Fuchsian analysis of singularities in Gowdy spacetimes beyond analyticity*, Class. Quant. Grav. **17**, 3305-3316 (2000)
- [49] A.D. Rendall: *Fuchsian methods and spacetime singularities*, Class. Quant. Grav. **21**, S295-S304 (2004)
- [50] H. Ringström: *Curvature blow up in Bianchi VIII and IX vacuum spacetimes*, Class. Quant. Grav. **17**, 713-731 (2000)
- [51] H. Ringström: *The Bianchi IX attractor*, Ann. H. Poincaré **2**, 405-500 (2001)
- [52] R. Steinbauer and J.A. Vickers: *The use of generalised functions and distributions in general relativity*, Class. Quant. Grav. **23**, R91-R114 (2006)
- [53] F. Ståhl: *Degeneracy of the b-boundary in general relativity*, Commun. Math. Phys. **208**, 331-353 (1999)
- [54] P. Szekeres: *On the singularities of a Riemannian manifold*, Publ. Mat. Debrecen **7**, 285-301 (1960)
- [55] K.P. Tod: *Isotropic cosmological singularities*, in *The Conformal Structure of Space-Time*, eds. J. Frauendiener and H. Friedrich, Lecture Notes in Physics, Volume 604, p. 123-134 (2002)
- [56] K.P. Tod: *Isotropic cosmological singularities: other matter models*, Class. Quant. Grav. **20**, 521-534 (2003)

- [57] K.P. Tod and C. Lübbe: *An extension theorem for conformal gauge singularities*, arXiv: 0710.5552 (2007)
- [58] K.P. Tod and C. Lübbe: *A global conformal extension theorem for perfect fluid Bianchi space-times*, arXiv: 0710.5723 (2007)
- [59] J.A. Vickers and J.P. Wilson: *Generalised hyperbolicity in conical space-times*, Class. Quant. Grav. **17**, 1333-1360 (2000)
- [60] J.A. Vickers and J.P. Wilson: *Generalised hyperbolicity: hypersurface singularities*, arXiv:gr-qc/0101018 (2001)
- [61] R.M. Wald: *General relativity*, University of Chicago Press, Chicago (1984)
- [62] M. Walker: *Block diagrams and the extension of timelike two-surfaces*, J. Math. Phys. **11**, 2280-2286 (1970)
- [63] H. Whitney: *Functions differentiable on the boundaries of regions*, Ann. Math. **35**, 482-485 (1934)
- [64] H. Whitney: *Analytic extensions of differentiable functions defined in closed sets*, Trans. Am. Math. Soc. **36**, 63-89 (1934)